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On some properties of k-circulant matrices with the generalized Pell-Padovan numbers

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Abstract. In this paper, we investigate the properties of the k-circulant matrix generated by the generalized Pell–Padovan numbers. We derive explicit formulas for the sum of entries, the maximum column sum norm ($\|\cdot\|_1$), the maximum row sum norm ($\|\cdot\|_\infty$), the Frobenius (Euclidean) norm ($\|\cdot\|_F$), as well as the eigenvalues and determinant of this matrix. Furthermore, we establish upper and lower bounds for its spectral norm ($\|\cdot\|_2$), thereby providing a comprehensive analysis of the structural and spectral characteristics of the k-circulant matrix associated with the generalized Pell–Padovan sequence.

Keywords: Pell-Padovan numbers, circulant matrix, k-circulant matrix, Tribonacci numbers, norm, spectral norm, matrix norms, determinant.

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1 Introduction

First, we recall definitions and some properties of the generalized Pell-Padovan sequence. A generalized Pell-Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-2} + V_{n-3}, \tag{1.1}$$

with initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$, not all zero, where c_0, c_1, c_2 are real or complex numbers.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -2V_{-(n-1)} + V_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$ Hence, recurrence (1.1) holds for all integers n.

The Binet formula for the generalized Pell-Padovan numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

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where

$$b_1 = V_2 - (\beta + \gamma)V_1 + \beta\gamma V_0, \quad b_2 = V_2 - (\alpha + \gamma)V_1 + \alpha\gamma V_0, \quad b_3 = V_2 - (\alpha + \beta)V_1 + \alpha\beta V_0. \tag{1.2}$$

Here, α, β, and γ are the roots of the cubic equation $x^3 - 2x - 1 = 0$. Moreover,

$$\begin{aligned} \alpha &= \frac{1+\sqrt{5}}{2}, \\ \beta &= \frac{1-\sqrt{5}}{2}, \\ \gamma &= -1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -2, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

Now we define four special cases of the sequence $\{V_n\}$. The Pell-Padovan sequence $\{R_n\}_{n\geq 0}$, the Pell-Perrin sequence $\{C_n\}_{n\geq 0}$, third-order Fibonacci-Pell sequence $\{G_n\}_{n\geq 0}$, and third-order Lucas-Pell sequence $\{B_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} R_{n+3} &= 2R_{n+1} + R_n, \quad R_0 = 1, R_1 = 1, R_2 = 1, \\ C_{n+3} &= 2C_{n+1} + C_n, \quad C_0 = 3, C_1 = 0, C_2 = 2, \\ G_{n+3} &= 2G_{n+1} + G_n, \quad G_0 = 1, G_1 = 0, G_2 = 2, \\ B_{n+3} &= 2B_{n+1} + B_n, \quad B_0 = 3, B_1 = 0, B_2 = 4. \end{aligned}$$

The sequences $\{R_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, $\{G_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} R_{-n} &= -2R_{-(n-1)} + R_{-(n-3)}, \\ C_{-n} &= -2C_{-(n-1)} + C_{-(n-3)}, \\ G_{-n} &= -2G_{-(n-1)} + G_{-(n-3)}, \\ B_{-n} &= -2B_{-(n-1)} + B_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively.

For more information on the Pell-Padovan sequence, see Soykan [20].

Theorem 1.1 gives a sum formula for generalized Pell-Padovan numbers.

Theorem 1.1. *Let x be a nonzero real or complex number. For $n \geq 0$, we have the following formula: If $x^3 + 2x^2 - 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k V_k = \frac{\Theta_1(x)}{\Theta(x)}.$$

where $\Theta_1(x) = x^{n+3}V_{n+3} + x^{n+2}V_{n+2} - (2x^2-1)x^{n+1}V_{n+1} - x^2V_2 - xV_1 + (2x^2-1)V_0$, $\Theta(x) = x^3 + 2x^2 - 1$.

Proof. This result follows directly from Theorem 2.1 (a) of [23] by setting the recurrence parameters to $r = 0, s = 2, t = 1$. □

Theorem 1.2 shows sum formulas for generalized Pell-Padovan numbers.

Theorem 1.2. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{i=0}^{n} V_i = \frac{1}{2}(V_{n+3} + V_{n+2} - V_{n+1} - V_2 - V_1 + V_0) = \dfrac{\Theta_1}{\Theta}$.

(b) $\sum_{i=0}^{n} iV_i = \frac{1}{4}((2n-1)V_{n+3} + (2n-3)V_{n+2} - (2n+3)V_{n+1} + 3V_2 + 5V_1 + V_0) = \dfrac{\Psi_1}{\Psi}$.

(c) $\sum_{i=0}^{n} V_i^2 = \frac{1}{2}((2n+11)\,V_{n+3}^2 + (2n+9)\,V_{n+2}^2 + (2n+11)\,V_{n+1}^2 - 4\,(n+5)\,V_{n+3}V_{n+2} - 4\,(n+6)\,V_{n+3}V_{n+1} + 4\,(n+6)\,V_{n+2}V_{n+1} - 9V_2^2 - 7V_1^2 - 9V_0^2 + 16V_2V_1 + 20V_2V_0 - 20V_1V_0) = \dfrac{\Delta_1}{\Delta}$.

(d) $\sum_{i=0}^{n} iV_i^2 = \frac{1}{4}((2n^2+18n+69)V_{n+3}^2 + (2n^2+14n+53)V_{n+2}^2 + (2n^2+18n+85)V_{n+1}^2 - 4(n^2+8n+31)V_{n+3}V_{n+2} - 4(n^2+10n+40)V_{n+3}V_{n+1} + 4(n^2+10n+38)V_{n+2}V_{n+1} - 53V_2^2 - 41V_1^2 - 69V_0^2 - 116V_1V_0 + 124V_2V_0 + 96V_2V_1) = \dfrac{\Omega_1}{\Omega}$.

Proof. **(a)** Take $x = 1, r = 0, s = 2, t = 1$ in [23], Theorem 2.1 (a) or take $r = 0, s = 2, t = 1$ in [24], Theorem 2.1 (a).

(b) Take $x = 1, r = 0, s = 2, t = 1$ in [26], Theorem 2.1 (a) or take $r = 0, s = 2, t = 1$ in [27], Theorem 2.1 (a).

(c) Take $x = 1, r = 0, s = 2, t = 1$ in [25], Theorem 3.1 (a) and use L'Hospital's rule, see [25], Theorem 4.2.

(d) Take $x = 1, r = 0, s = 2, t = 1$ in [28], Theorem 2.1 (a) and use L'Hospital's rule, see [28], Theorem 3.1.

□

Using the recurrence relation $V_{n+3} = 2V_{n+1} + V_n$, we can write Theorem 1.2 as follows.

Theorem 1.3. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{i=0}^{n} V_i = \frac{1}{2}(V_{n+2} + V_{n+1} + V_n - V_2 - V_1 + V_0) = \dfrac{\Theta_1}{\Theta}$.

(b) $\sum_{i=0}^{n} iV_i = \frac{1}{4}((2n-3)V_{n+2} + (2n-5)V_{n+1} + (2n-1)V_n + 3V_2 + 5V_1 + V_0) = \dfrac{\Psi_1}{\Psi}$.

(c) $\sum_{i=0}^{n} V_i^2 = \frac{1}{2}((2n+9)V_{n+2}^2 + (2n+7)V_{n+1}^2 + (2n+11)V_n^2 - 4(n+4)V_{n+2}V_{n+1} - 4(n+5)V_{n+2}V_n + 4(n+5)V_{n+1}V_n - 9V_2^2 - 7V_1^2 - 9V_0^2 + 16V_2V_1 + 20V_2V_0 - 20V_1V_0) = \dfrac{\Delta_1}{\Delta}$.

(d) $\sum_{i=0}^{n} iV_i^2 = \frac{1}{4}((2n^2+14n+53)V_{n+2}^2 + (2n^2+10n+41)V_{n+1}^2 + (2n^2+18n+69)V_n^2 - 4(n^2+6n+24)V_{n+2}V_{n+1} - 4(n^2+8n+31)V_{n+2}V_n + 4(n^2+8n+29)V_{n+1}V_n - 53V_2^2 - 41V_1^2 - 69V_0^2 - 116V_1V_0 + 124V_2V_0 + 96V_2V_1) = \dfrac{\Omega_1}{\Omega}$.

From Theorem 1.3, we obtain the following corollary, which gives sum formulas for Pell-Padovan numbers (take $V_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 1.4. *For $n \geq 0$, Pell-Padovan numbers have the following properties:*

(a) $\sum_{i=0}^{n} R_i = \frac{1}{2}(R_{n+2} + R_{n+1} + R_n - 1)$.

(b) $\sum_{i=0}^{n} iR_i = \frac{1}{4}((2n-3)R_{n+2} + (2n-5)R_{n+1} + (2n-1)R_n + 9)$.

(c) $\sum_{i=0}^{n} R_i^2 = \frac{1}{2}((2n+9)R_{n+2}^2 + (2n+7)R_{n+1}^2 + (2n+11)R_n^2 - 4(n+4)R_{n+2}R_{n+1} - 4(n+5)R_{n+2}R_n + 4(n+5)R_{n+1}R_n - 9)$.

(d) $\sum_{i=0}^{n} iR_i^2 = \frac{1}{4}((2n^2+14n+53)R_{n+2}^2 + (2n^2+10n+41)R_{n+1}^2 + (2n^2+18n+69)R_n^2 - 4(n^2+6n+24)R_{n+2}R_{n+1} - 4(n^2+8n+31)R_{n+2}R_n + 4(n^2+8n+29)R_{n+1}R_n - 59)$.

Taking $V_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in Theorem 1.3, we get the following corollary, which gives sum formulas for Pell-Perrin numbers.

Corollary 1.5. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

(a) $\sum_{i=0}^{n} C_i = \frac{1}{2}(C_{n+2} + C_{n+1} + C_n + 1)$.

(b) $\sum_{i=0}^{n} iC_i = \frac{1}{4}((2n-3)C_{n+2} + (2n-5)C_{n+1} + (2n-1)C_n + 9)$.

(c) $\sum_{i=0}^{n} C_i^2 = \frac{1}{2}((2n+9)C_{n+2}^2 + (2n+7)C_{n+1}^2 + (2n+11)C_n^2 - 4(n+4)C_{n+2}C_{n+1} - 4(n+5)C_{n+2}C_n + 4(n+5)C_{n+1}C_n + 3)$.

(d) $\sum_{i=0}^{n} iC_i^2 = \frac{1}{4}((2n^2+14n+53)C_{n+2}^2 + (2n^2+10n+41)C_{n+1}^2 + (2n^2+18n+69)C_n^2 - 4(n^2+6n+24)C_{n+2}C_{n+1} - 4(n^2+8n+31)C_{n+2}C_n + 4(n^2+8n+29)C_{n+1}C_n - 89)$.

From Theorem 1.3, we have the following corollary, which presents sum formulas for third-order Fibonacci-Pell numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 1.6. *For $n \geq 0$, third-order Fibonacci-Pell numbers have the following properties:*

(a) $\sum_{i=0}^{n} G_i = \frac{1}{2}(G_{n+2} + G_{n+1} + G_n - 1)$.

(b) $\sum_{i=0}^{n} iG_i = \frac{1}{4}((2n-3)G_{n+2} + (2n-5)G_{n+1} + (2n-1)G_n + 7)$.

(c) $\sum_{i=0}^{n} G_i^2 = \frac{1}{2}((2n+9)G_{n+2}^2 + (2n+7)G_{n+1}^2 + (2n+11)G_n^2 - 4(n+4)G_{n+2}G_{n+1} - 4(n+5)G_{n+2}G_n + 4(n+5)G_{n+1}G_n - 5)$.

(d) $\sum_{i=0}^{n} iG_i^2 = \frac{1}{4}((2n^2+14n+53)G_{n+2}^2 + (2n^2+10n+41)G_{n+1}^2 + (2n^2+18n+69)G_n^2 - 4(n^2+6n+24)G_{n+2}G_{n+1} - 4(n^2+8n+31)G_{n+2}G_n + 4(n^2+8n+29)G_{n+1}G_n - 33)$.

Taking $V_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in Theorem 1.3, we obtain the following corollary, which shows sum formulas for third-order Lucas-Pell numbers.

Corollary 1.7. *For $n \geq 0$, third-order Lucas-Pell numbers have the following properties:*

(a) $\sum_{i=0}^{n} B_i = \frac{1}{2}(B_{n+2} + B_{n+1} + B_n - 1)$.

(b) $\sum_{i=0}^{n} iB_i = \frac{1}{4}((2n-3)B_{n+2} + (2n-5)B_{n+1} + (2n-1)B_n + 15)$.

(c) $\sum_{i=0}^{n} B_i^2 = \frac{1}{2}((2n+9)B_{n+2}^2 + (2n+7)B_{n+1}^2 + (2n+11)B_n^2 - 4(n+4)B_{n+2}B_{n+1} - 4(n+5)B_{n+2}B_n + 4(n+5)B_{n+1}B_n + 15)$.

(d) $\sum_{i=0}^{n} iB_i^2 = \frac{1}{4}((2n^2+14n+53)B_{n+2}^2 + (2n^2+10n+41)B_{n+1}^2 + (2n^2+18n+69)B_n^2 - 4(n^2+6n+24)B_{n+2}B_{n+1} - 4(n^2+8n+31)B_{n+2}B_n + 4(n^2+8n+29)B_{n+1}B_n + 19)$.

2 Main results

In this section, we recall some information on k-circulant matrices, Frobenius norm, spectral norm, maximum column Euclidean norm, and maximum row Euclidean norm. Let $n \geq 2$ be an integer and k be any real or complex number. An $n \times n$ matrix $C_k = \left(c_{ij}\right) \in M_{n\times n}(\mathbb{C})$ is called a k-circulant matrix if it is of the form

$$C_k = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

The k-circulant matrix C_k is denoted by $C_k = \mathrm{Circ}_k(c_0, c_1, \dots, c_{n-1})$.

If $k = 1$, then the 1-circulant matrix is called a circulant matrix and is denoted by $C = \mathrm{Circ}(c_0, c_1, \dots, c_{n-1})$. The circulant matrix was first proposed by Davis in [4].

The Frobenius (or Euclidean) norm and spectral norm of an $m \times n$ matrix $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1\leq i\leq n}|\lambda_i(A^*A)|\right)^{1/2},$$

where $\lambda_i(A^*A)$ are the eigenvalues of the matrix A^*A and A^* is the conjugate transpose of the matrix A. The following inequality holds for any matrix $A = (a_{ij})_{n\times n} \in M_{n\times n}(\mathbb{C})$ (see [33], Theorem 1 and Table 1):

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F. \tag{2.1}$$

Hence

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2.$$

In the literature, there are other types of matrix norms. The maximum column sum matrix norm of an $n \times n$ matrix $A = \left(a_{ij}\right)$ is $\|A\|_1 = \max\limits_{1\leq j\leq n}\sum_{i=1}^{n}\left|a_{ij}\right|$, and the maximum row sum matrix norm is $\|A\|_\infty = \max\limits_{1\leq i\leq n}\sum_{j=1}^{n}\left|a_{ij}\right|$. The maximum column length norm $c_1(A)$ and the maximum row length norm $r_1(A)$ of an $m \times n$ matrix $A = \left(a_{ij}\right)$ are defined as follows:

$$c_1(A) = \max_{1\leq j\leq n}\left(\sum_{i=1}^{n}\left|a_{ij}\right|^2\right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2}.$$

There is a relation between $\|\cdot\|_2$, $c_1(\cdot)$, and $r_1(\cdot)$ norms as follows:

Lemma 2.1. *[7] For any matrices $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ and $B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$, we have*

$$\|A \circ B\|_2 \leq r_1(A)c_1(B).$$

and

$$\|A \circ B\|_2 \leq \|A\|_2\,\|B\|_2.$$

and

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2 .$$

where $A \circ B$ *is the Hadamard product, defined by*

$$A \circ B = (a_{ij} b_{ij}),$$

and $A \otimes B$ *is the Kronecker product, defined by*

$$A \otimes B = (a_{ij} B).$$

For more details on matrix norms, see, for example, [8]. In the following Table 1, we present a few special studies on the Frobenius norm, spectral norm, maximum column length norm, and maximum row length norm of circulant (k-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences, which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Table 1. Related works on norms of circulant-type matrices with sequences.

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[5,6,15]
Pell, Pell-Lucas	[1,29]
Jacobsthal, Jacobsthal-Lucas	[16,30–32]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[17,18]
Padovan, Perrin	[3,14,19]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[13]

Here, we need the following two lemmas for our calculations.

Lemma 2.2. *[[2], Lemma 4] Let* $C_k = \mathrm{Circ}_k(c_0, c_1, \ldots, c_{n-1})$ *be an* $n \times n$ *k-circulant matrix. Then we have*

$$\lambda_j(C_k) = \sum_{p=0}^{n-1} k^{\frac{p}{n}} \omega^{-jp} c_p = \sum_{p=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^p c_p,$$

where $\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}$, $j = 0, 1, 2, \ldots, n-1$. *Moreover, in this case*

$$c_p = \frac{1}{n} \sum_{j=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^{-p} \lambda_j(C_k), \quad p = 0, 1, 2, \ldots, n-1,$$

where $k^{\frac{1}{n}}$ *denotes the principal nth root.*

Lemma 2.3. *[8] Let* A *be an* $n \times n$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$*. Then,* A *is a normal matrix if and only if the eigenvalues of* AA^* *are* $|\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2, \ldots, |\lambda_n|^2$*, where* A^* *is the conjugate transpose of the matrix* A*.*

We now define the k-circulant matrix with generalized Pell-Padovan number entries. Throughout this paper, the k-circulant matrix, whose entries are the generalized Pell-Padovan numbers, will be denoted by $C_n(V)_k = \mathrm{Circ}_k(V_0, V_1, \ldots, V_{n-1})$.

Definition 2.4. An $n \times n$ k-circulant matrix with generalized Pell-Padovan number entries is defined by

$$C_n(V)_k = \text{Circ}_k(V_0, V_1, \ldots, V_{n-1}) = \begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ kV_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n\times n}. \quad (2.2)$$

We call this matrix the generalized Pell-Padovan k-circulant matrix. We handle two special cases of the generalized Pell-Padovan k-circulant matrix, namely the Pell-Padovan k-circulant matrix: $C_n(R)_k = \text{Circ}_k(R_0, R_1, \ldots, R_{n-1})$, the Pell-Perrin k-circulant matrix: $C_n(C)_k = \text{Circ}_k(C_0, C_1, \ldots, C_{n-1})$, the third-order Fibonacci-Pell k-circulant matrix: $C_n(G)_k = \text{Circ}_k(G_0, G_1, \ldots, G_{n-1})$, and the third-order Lucas-Pell k-circulant matrix: $C_n(B)_k = \text{Circ}_k(B_0, B_1, \ldots, B_{n-1})$.

Now, we denote the sum of entries of $C_n(V)_k$ as $S(C_n(V)_k)$.

Lemma 2.5. *The sum of entries of $C_n(V)_k$ is*
$S(C_n(V)_k) = \frac{1}{4}((2kn - 3k + 3)V_{n+2} + (2kn - 5k + 5)V_{n+1} - (k + 2kn - 1)V_n + (3k - 2n - 3)V_2 + (5k - 2n - 5)V_1 + (2n + k - 1)V_0)$.

Proof. From the definition of $C_n(V)_k$, using Theorem 1.3, we obtain

$$\begin{aligned} S(C_n(V)_k) &= nV_0 + ((n-1)+k)V_1 + ((n-2)+2k)V_2 + \cdots + (1+(n-1)k)V_{n-1} \\ &= \sum_{i=0}^{n-1}(n-i)V_i + k\sum_{i=1}^{n-1} iV_i \\ &= n\sum_{i=0}^{n-1} V_i + (k-1)\sum_{i=1}^{n-1} iV_i \\ &= \frac{1}{4}((2kn-3k+3)V_{n+2} + (2kn-5k+5)V_{n+1} - (k+2kn-1)V_n \\ &\quad +(3k-2n-3)V_2 + (5k-2n-5)V_1 + (2n+k-1)V_0). \end{aligned}$$

□

Taking $V_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$, $V_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$, $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$, and $V_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$, respectively, in the last lemma, we obtain the following corollary.

Corollary 2.6. *We have the following results:*

(a) *The sum of entries of $C_n(R)_k$ is*

$S(C_n(R)_k) = \frac{1}{4}((2kn - 3k + 3)R_{n+2} + (2kn - 5k + 5)R_{n+1} - (k + 2kn - 1)R_n + (9k - 2n - 9))$.

(b) *The sum of entries of $C_n(C)_k$ is*

$S(C_n(C)_k) = \frac{1}{4}((2kn - 3k + 3)C_{n+2} + (2kn - 5k + 5)C_{n+1} - (k + 2kn - 1)C_n + (9k + 2n - 9))$.

(c) *The sum of entries of $C_n(G)_k$ is*

$S(C_n(G)_k) = \frac{1}{4}((2kn - 3k + 3)G_{n+2} + (2kn - 5k + 5)G_{n+1} - (k + 2kn - 1)G_n + (7k - 2n - 7))$.

(d) *The sum of entries of* $C_n(B)_k$ *is*

$S(C_n(B)_k) = \frac{1}{4}((2kn-3k+3)B_{n+2} + (2kn-5k+5)B_{n+1} - (k+2kn-1)B_n + (15k-2n-15))$.

Next, we give the maximum column sum matrix norm $\|C_n(V)_k\|_1$ and the maximum row sum matrix norm $\|C_n(V)_k\|_\infty$ of the matrix $C_n(V)_k = (c_{ij})$ under certain conditions on the generalized Pell-Padovan sequence V_n and k.

Theorem 2.7. *Assume* $V_p \geq 0$ *for all nonnegative integers* p. *Then we have the following formulas: If* $k \geq 1$, *then*

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = \frac{1}{2}(kV_{n+2} + kV_{n+1} - kV_n - kV_2 - kV_1 + (2-k)V_0),$$

and if $k < 1$, *then*

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = \frac{1}{2}(V_{n+2} + V_{n+1} - V_n - V_2 - V_1 + V_0).$$

Proof. Suppose that $k \geq 1$. Then from the definition of the matrix $C_n(V)_k = (c_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}
\|C_n(V)_k\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |c_{ij}| = \max_{1\leq j\leq n} \{|c_{1j}| + |c_{2j}| + |c_{3j}| + \cdots + |c_{nj}|\} \\
&= |c_{11}| + |c_{21}| + |c_{31}| + \cdots + |c_{n1}| \\
&= V_0 + kV_{n-1} + kV_{n-2} + \cdots + kV_3 + kV_2 + kV_1 \\
&= (V_0 - kV_0 - kV_n) + k\sum_{i=0}^{n} V_i \\
&= \frac{1}{2}(kV_{n+2} + kV_{n+1} - kV_n - kV_2 - kV_1 + (2-k)V_0).
\end{aligned}$$

Similarly, we obtain

$$\|C_n(V)_k\|_\infty = \frac{1}{2}(kV_{n+2} + kV_{n+1} - kV_n - kV_2 - kV_1 + (2-k)V_0).$$

Suppose now that $k < 1$. Then from the definition of the matrix $C_n(V)_k = (c_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}
\|C_n(V)_k\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |c_{ij}| = \max_{1\leq j\leq n} \{|c_{1j}| + |c_{2j}| + |c_{3j}| + \cdots + |c_{nj}|\} \\
&= |c_{1n}| + |c_{2n}| + |c_{3n}| + \cdots + |c_{nn}| \\
&= V_{n-1} + V_{n-2} + \cdots + V_3 + V_2 + V_1 + V_0 \\
&= -V_n + \sum_{i=0}^{n} V_i \\
&= \frac{1}{2}(V_{n+2} + V_{n+1} - V_n - V_2 - V_1 + V_0).
\end{aligned}$$

Similarly, we get

$$\|C_n(V)_k\|_\infty = \frac{1}{2}(V_{n+2} + V_{n+1} - V_n - V_2 - V_1 + V_0).$$

□

Substituting $V_n = R_n$ (with $R_0 = 1, R_1 = 1, R_2 = 1$), $V_n = C_n$ (with $C_0 = 3, C_1 = 0, C_2 = 2$), $V_n = G_n$ (with $G_0 = 1, G_1 = 0, G_2 = 2$), and $V_n = B_n$ (with $B_0 = 3, B_1 = 0, B_2 = 4$) into Theorem 2.7, we obtain the following corollary.

Corollary 2.8. *We get the following results:*

(a) *If $k \geq 1$, then*

$$\|C_n(R)_k\|_1 = \|C_n(R)_k\|_\infty = \frac{1}{2}(kR_{n+2} + kR_{n+1} - kR_n + (2-3k)),$$

and if $k < 1$, then

$$\|C_n(R)_k\|_1 = \|C_n(R)_k\|_\infty = \frac{1}{2}(R_{n+2} + R_{n+1} - R_n - 1).$$

(b) *If $k \geq 1$, then*

$$\|C_n(C)_k\|_1 = \|C_n(C)_k\|_\infty = \frac{1}{2}(kC_{n+2} + kC_{n+1} - kC_n + (6-5k)),$$

and if $k < 1$, then

$$\|C_n(C)_k\|_1 = \|C_n(C)_k\|_\infty = \frac{1}{2}(C_{n+2} + C_{n+1} - C_n + 1).$$

(c) *If $k \geq 1$, then*

$$\|C_n(G)_k\|_1 = \|C_n(G)_k\|_\infty = \frac{1}{2}(kG_{n+2} + kG_{n+1} - kG_n + (2-3k)),$$

and if $k < 1$, then

$$\|C_n(G)_k\|_1 = \|C_n(G)_k\|_\infty = \frac{1}{2}(G_{n+2} + G_{n+1} - G_n - 1).$$

(d) *If $k \geq 1$, then*

$$\|C_n(B)_k\|_1 = \|C_n(B)_k\|_\infty = \frac{1}{2}(kB_{n+2} + kB_{n+1} - kB_n + (6-7k)),$$

and if $k < 1$, then

$$\|C_n(B)_k\|_1 = \|C_n(B)_k\|_\infty = \frac{1}{2}(B_{n+2} + B_{n+1} - B_n - 1).$$

Here, we determine the Euclidean (Frobenius) norm of the k-circulant matrix $C_n(V)_k$.

Theorem 2.9. *The Euclidean (Frobenius) norm of the k-circulant matrix $C_n(V)_k$ is:*

$$\|C_n(V)_k\|_F = \sqrt{n\,(\varphi_1(V)) + \varphi_2(V)},$$

where $\varphi_1(V) = \frac{1}{2}((2n+9)V_{n+2}^2 + (2n+7)V_{n+1}^2 + (2n+9)V_n^2 - 4(n+4)V_{n+2}V_{n+1} - 4(n+5)V_{n+2}V_n + 4(n+5)V_{n+1}V_n - 9V_2^2 - 7V_1^2 - 9V_0^2 + 16V_2V_1 + 20V_2V_0 - 20V_1V_0)$,

$\varphi_2(V) = \frac{1}{4}(|k|^2 - 1)((2n^2 + 14n + 53)V_{n+2}^2 + (2n^2 + 10n + 41)V_{n+1}^2 + (2n^2 + 14n + 69)V_n^2 - 4(n^2 + 6n + 24)V_{n+2}V_{n+1} - 4(n^2 + 8n + 31)V_{n+2}V_n + 4(n^2 + 8n + 29)V_{n+1}V_n - 53V_2^2 - 41V_1^2 - 69V_0^2 - 116V_1V_0 + 124V_2V_0 + 96V_2V_1)$.

Proof. From the definition of the Euclidean norm of a matrix, using Theorem 1.3, we obtain

$$\begin{aligned}
(\|C_n(V)_k\|_F)^2 &= \sum_{i=1,j=1}^{n} |c_{ij}|^2 \\
&= \sum_{i=0}^{n-1} (n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\
&= n\sum_{i=0}^{n-1} V_i^2 + (|k|^2-1)\sum_{i=1}^{n-1} iV_i^2 \\
&= n\,(\varphi_1(V)) + \varphi_2(V),
\end{aligned}$$

where $\varphi_1(V)$ and $\varphi_2(V)$ are as in the statement of the theorem. Next, it follows that

$$\|C_n(V)_k\|_F = \sqrt{n\,(\varphi_1(V)) + \varphi_2(V)}.$$

□

Note that

$$\varphi_1(V) = \sum_{i=0}^{n-1} V_i^2,$$

and

$$\varphi_2(V) = (|k|^2-1)\sum_{i=1}^{n-1} iV_i^2.$$

Taking $V_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$, $V_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$, $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$, and $V_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$, respectively, in Theorem 2.9, we get the following corollary.

Corollary 2.10. *We have the following results:*

(a) *The Euclidean (Frobenius) norm of the k-circulant matrix $C_n(R)_k$ is:*

$$\|C_n(R)_k\|_F = \sqrt{n\,(\varphi_1(R)) + \varphi_2(R)},$$

where

$\varphi_1(R) = \frac{1}{2}((2n+9)R_{n+2}^2 + (2n+7)R_{n+1}^2 + (2n+9)R_n^2 - 4(n+4)R_{n+2}R_{n+1} - 4(n+5)R_{n+2}R_n + 4(n+5)R_{n+1}R_n - 9)$,

$\varphi_2(R) = \frac{1}{4}(|k|^2-1)((2n^2+14n+53)R_{n+2}^2 + (2n^2+10n+41)R_{n+1}^2 + (2n^2+14n+69)R_n^2 - 4(n^2+6n+24)R_{n+2}R_{n+1} - 4(n^2+8n+31)R_{n+2}R_n + 4(n^2+8n+29)R_{n+1}R_n - 59)$.

(b) *The Euclidean (Frobenius) norm of the k-circulant matrix $C_n(C)_k$ is:*

$$\|C_n(C)_k\|_F = \sqrt{n\,(\varphi_1(C)) + \varphi_2(C)},$$

where

$\varphi_1(C) = \frac{1}{2}((2n+9)C_{n+2}^2 + (2n+7)C_{n+1}^2 + (2n+9)C_n^2 - 4(n+4)C_{n+2}C_{n+1} - 4(n+5)C_{n+2}C_n + 4(n+5)C_{n+1}C_n + 3)$,

$\varphi_2(C) = \frac{1}{4}(|k|^2-1)((2n^2+14n+53)C_{n+2}^2 + (2n^2+10n+41)C_{n+1}^2 + (2n^2+14n+69)C_n^2 - 4(n^2+6n+24)C_{n+2}C_{n+1} - 4(n^2+8n+31)C_{n+2}C_n + 4(n^2+8n+29)C_{n+1}C_n - 89)$.

(c) *The Euclidean (Frobenius) norm of the k-circulant matrix $C_n(G)_k$ is:*

$$\|C_n(G)_k\|_F = \sqrt{n\left(\varphi_1(G)\right) + \varphi_2(G)},$$

where
$\varphi_1(G) = \frac{1}{2}((2n+9)G_{n+2}^2 + (2n+7)G_{n+1}^2 + (2n+9)G_n^2 - 4(n+4)G_{n+2}G_{n+1} - 4(n+5)G_{n+2}G_n + 4(n+5)G_{n+1}G_n - 5)$,
$\varphi_2(G) = \frac{1}{4}(|k|^2-1)((2n^2+14n+53)G_{n+2}^2 + (2n^2+10n+41)G_{n+1}^2 + (2n^2+14n+69)G_n^2 - 4(n^2+6n+24)G_{n+2}G_{n+1} - 4(n^2+8n+31)G_{n+2}G_n + 4(n^2+8n+29)G_{n+1}G_n - 33)$.

(d) *The Euclidean (Frobenius) norm of the k-circulant matrix $C_n(B)_k$ is:*

$$\|C_n(B)_k\|_F = \sqrt{n\left(\varphi_1(B)\right) + \varphi_2(B)},$$

where
$\varphi_1(B) = \frac{1}{2}((2n+9)B_{n+2}^2 + (2n+7)B_{n+1}^2 + (2n+9)B_n^2 - 4(n+4)B_{n+2}B_{n+1} - 4(n+5)B_{n+2}B_n + 4(n+5)B_{n+1}B_n + 15)$,
$\varphi_2(B) = \frac{1}{4}(|k|^2-1)((2n^2+14n+53)B_{n+2}^2 + (2n^2+10n+41)B_{n+1}^2 + (2n^2+14n+69)B_n^2 - 4(n^2+6n+24)B_{n+2}B_{n+1} - 4(n^2+8n+31)B_{n+2}B_n + 4(n^2+8n+29)B_{n+1}B_n + 19)$.

The next theorem presents the eigenvalues of the matrix in (2.2).

Theorem 2.11. *The eigenvalues of $C_n(V)_k$ are*

$$\lambda_j(C_n(V)_k) = \frac{\Phi_j(V)}{(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

where
$\Phi_j(V) = kV_n - V_0 - k^{\frac{1}{n}}(-kV_{n+1} + V_1)\omega^{-j} + k^{\frac{2}{n}}(kV_{n+2} - 2kV_n - V_2 + 2V_0)\omega^{-2j}$,
and

$$\begin{aligned} \omega &= \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, \\ j &= 0,1,2,3,\ldots,n-1. \end{aligned}$$

Proof. By using Lemma 2.2, we have

$$\begin{aligned} \lambda_j(C_n(V)_k) &= \sum_{p=0}^{n-1} k^{\frac{p}{n}}\omega^{-jp}V_p \\ &= -k\omega^{-jn}V_n + \sum_{p=0}^{n} k^{\frac{p}{n}}\omega^{-jp}V_p \\ &= -k\omega^{-jn}V_n + \sum_{p=0}^{n} (k^{\frac{1}{n}}\omega^{-j})^p V_p. \end{aligned}$$

Now using Theorem 1.1 (by putting $x = k^{\frac{1}{n}}\omega^{-j}$) and the recurrence relation $V_{n+3} = 2V_{n+1} + V_n$, we get the required result. □

Taking $V_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$, $V_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$, $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$, and $V_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$, respectively, in Theorem 2.11, we obtain the following corollary.

Corollary 2.12. *We have the following results:*

(a) *The eigenvalues of $C_n(R)_k$ are*

$$\lambda_j(C_n(R)_k) = \frac{\Phi_j(R)}{(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(b) *the eigenvalues of $C_n(C)_k$ are*

$$\lambda_j(C_n(C)_k) = \frac{\Phi_j(C)}{(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(c) *the eigenvalues of $C_n(G)_k$ are*

$$\lambda_j(C_n(G)_k) = \frac{\Phi_j(G)}{(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(d) *the eigenvalues of $C_n(B)_k$ are*

$$\lambda_j(C_n(B)_k) = \frac{\Phi_j(B)}{(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

where

$\Phi_j(R) = kR_n - 1 - k^{\frac{1}{n}}(-kR_{n+1} + 1)\omega^{-j} + k^{\frac{2}{n}}(kR_{n+2} - 2kR_n + 1)\omega^{-2j}$,

$\Phi_j(C) = kC_n - 3 - k^{\frac{1}{n}}(-kC_{n+1})\omega^{-j} + k^{\frac{2}{n}}(kC_{n+2} - 2kC_n + 4)\omega^{-2j}$,

$\Phi_j(G) = kG_n - 1 - k^{\frac{1}{n}}(-kG_{n+1})\omega^{-j} + k^{\frac{2}{n}}(kG_{n+2} - 2kG_n)\omega^{-2j}$,

$\Phi_j(B) = kB_n - 3 - k^{\frac{1}{n}}(-kB_{n+1})\omega^{-j} + k^{\frac{2}{n}}(kB_{n+2} - 2kB_n + 2)\omega^{-2j}$,

$\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}$, $j = 0, 1, 2, 3, \ldots, n-1$.

The next theorem presents upper and lower bounds for the spectral norm of $C_n(V)_k$.

Theorem 2.13. *Let $C_n(V)_k = \mathrm{Circ}_k(V_0, V_1, \ldots, V_{n-1})$ be a k-circulant matrix. Then if $|k| \geq 1$, then*

$$\sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{V_0^2 + |k|^2\left(-V_0^2 + \varphi_1(V)\right)}\sqrt{1 - V_0^2 + \varphi_1(V)},$$

and if $|k| < 1$, then

$$|k|\sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{n\left(\varphi_1(V)\right)},$$

where $\varphi_1(V)$ is as in Theorem 2.9.

Proof. Note that we can write $\varphi_1(V)$ in the following forms.

$$\begin{aligned}
\varphi_1(V) &= \sum_{i=0}^{n-1} V_i^2 \\
&= \frac{1}{2}((2n+9)V_{n+2}^2 + (2n+7)V_{n+1}^2 + (2n+9)V_n^2 - 4(n+4)V_{n+2}V_{n+1} - 4(n+5)V_{n+2}V_n \\
&\quad +4(n+5)V_{n+1}V_n - 9V_2^2 - 7V_1^2 - 9V_0^2 + 16V_2V_1 + 20V_2V_0 - 20V_1V_0), \\
\varphi_1(V) &= V_0^2 + \sum_{i=1}^{n-1} V_i^2 \Rightarrow -V_0^2 + \varphi_1(V) = \sum_{i=1}^{n-1} V_i^2.
\end{aligned}$$

From Theorem 2.9, we know that the Euclidean (Frobenius) norm of the k-circulant matrix $C_n(V)_k$ is

$$\begin{aligned}(\|C_n(V)_k\|_F)^2 &= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2\sum_{i=1}^{n-1} iV_i^2 \\ &= n\sum_{i=0}^{n-1} V_i^2 + (|k|^2-1)\sum_{i=1}^{n-1} iV_i^2.\end{aligned}$$

If $|k| \geq 1$, then we have, using Theorem 1.3,

$$(\|C_n(V)_k\|_F)^2 \geq \sum_{i=0}^{n-1}(n-i)V_i^2 + \sum_{i=1}^{n-1} iV_i^2 = n\sum_{i=0}^{n-1} V_i^2 = n\left(\varphi_1(V)\right).$$

i.e.

$$\|C_n(V)_k\|_F \geq \sqrt{n\left(\varphi_1(V)\right)}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq \sqrt{\varphi_1(V)}.$$

Then by (2.1), we obtain

$$\|C_n(V)_k\|_2 \geq \sqrt{\varphi_1(V)}.$$

Similarly, if $|k| < 1$, then we get

$$\begin{aligned}\|C_n(V)_k\|_F^2 &= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2\sum_{i=1}^{n-1} iV_i^2 \\ &\geq \sum_{i=0}^{n-1}(n-i)\,|k|^2\,V_i^2 + |k|^2\sum_{i=1}^{n-1} iV_i^2 = n\,|k|^2\sum_{i=0}^{n-1} V_i^2 \\ &= n\,|k|^2\left(\varphi_1(V)\right).\end{aligned}$$

i.e.,

$$\|C_n(V)_k\|_F \geq \sqrt{n\,|k|^2\left(\varphi_1(V)\right)}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq |k|\,\sqrt{\varphi_1(V)}.$$

Then by considering (2.1), we have

$$\|C_n(V)_k\|_2 \geq |k|\,\sqrt{(\varphi_1(V))}.$$

Here, for $|k| \geq 1$, we give an upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. Let the matrices B and C be as

$$B = \begin{pmatrix} V_0 & 1 & 1 & \cdots & 1 & 1 \\ kV_{n-1} & V_0 & 1 & \cdots & 1 & 1 \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n\times n},$$

and

$$C=\begin{pmatrix} 1 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ 1 & 1 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ 1 & 1 & 1 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{n\times n},$$

such that $C_n(V)_k = B \circ C$. Then, we have

$$r_1(B)=\max_{1\le i\le n}\left(\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2}=\sqrt{V_0^2+|k|^2\sum_{j=1}^{n-1}V_j^2}=\sqrt{V_0^2+|k|^2\left(-V_0^2+\varphi_1(V)\right)},$$

$$c_1(C)=\max_{1\le j\le n}\left(\sum_{i=1}^{n}|c_{ij}|^2\right)^{1/2}=\sqrt{1+\sum_{i=1}^{n-1}V_i^2}=\sqrt{1-V_0^2+\varphi_1(V)}.$$

By Lemma 2.1, we have

$$\|C_n(V)_k\|_2\le r_1(B)c_1(C)=\sqrt{V_0^2+|k|^2\left(-V_0^2+\varphi_1(V)\right)}\sqrt{1-V_0^2+\varphi_1(V)}.$$

For $|k|<1$, we present an upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. We define the matrices D and E as

$$D=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ k & 1 & 1 & \cdots & 1 & 1 \\ k & k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \cdots & k & 1 \end{pmatrix}_{n\times n},$$

and

$$E=\begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ V_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ V_1 & V_2 & V_3 & \cdots & V_{n-1} & V_0 \end{pmatrix}_{n\times n},$$

such that $C_n(V)_k = D \circ E$. Then we have

$$r_1(D)=\max_{1\le i\le n}\left(\sum_{j=1}^{n}|d_{ij}|^2\right)^{1/2}=\sqrt{n},$$

and

$$c_1(E)=\max_{1\le j\le n}\left(\sum_{i=1}^{n}|e_{ij}|^2\right)^{1/2}=\sqrt{\sum_{i=0}^{n-1}V_i^2}=\sqrt{\varphi_1(V)}.$$

By Lemma 2.1, we have

$$\|C_n(V)_k\|_2\le r_1(D)c_1(E)=\sqrt{n\left(\varphi_1(V)\right)}.$$

So, the proof is completed as desired. $\square$

We consider four special cases of Theorem 2.13.

Firstly, the following corollary presents the upper and lower bounds of the spectral norm of $C_n(R)_k$.

Corollary 2.14. *Let* $C_n(R)_k = \mathrm{Circ}_k(R_0, R_1, \ldots, R_{n-1})$ *be a Pell-Padovan k-circulant matrix. Then if* $|k| \geq 1$, *then*

$$\sqrt{\varphi_1(R)} \leq \|C_n(R)_k\|_2 \leq \sqrt{R_0^2 + |k|^2\left(-R_0^2 + \varphi_1(R)\right)}\sqrt{1 - R_0^2 + \varphi_1(R)},$$

and if $|k| < 1$, *then*

$$|k|\sqrt{\varphi_1(R)} \leq \|C_n(R)_k\|_2 \leq \sqrt{n\left(\varphi_1(R)\right)},$$

where $\varphi_1(R)$ *is as in Corollary 2.10.*

Proof. Take $V_n = R_n,\ R_0 = 1, R_1 = 1, R_2 = 1$ in Theorem 2.13. □

Secondly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(C)_k$.

Corollary 2.15. *Let* $C_n(C)_k = \mathrm{Circ}_k(C_0, C_1, \ldots, C_{n-1})$ *be a Pell-Perrin k-circulant matrix. Then if* $|k| \geq 1$, *then*

$$\sqrt{\varphi_1(C)} \leq \|C_n(C)_k\|_2 \leq \sqrt{C_0^2 + |k|^2\left(-C_0^2 + \varphi_1(C)\right)}\sqrt{1 - C_0^2 + \varphi_1(C)},$$

and if $|k| < 1$, *then*

$$|k|\sqrt{\varphi_1(C)} \leq \|C_n(C)_k\|_2 \leq \sqrt{n\left(\varphi_1(C)\right)},$$

where $\varphi_1(C)$ *is as in Corollary 2.10.*

Proof. Take $V_n = C_n,\ C_0 = 3, C_1 = 0, C_2 = 2$ in Theorem 2.13. □

Thirdly, the following corollary presents the upper and lower bounds of the spectral norm of $C_n(G)_k$.

Corollary 2.16. *Let* $C_n(G)_k = \mathrm{Circ}_k(G_0, G_1, \ldots, G_{n-1})$ *be a third-order Fibonacci-Pell k-circulant matrix. Then if* $|k| \geq 1$, *then*

$$\sqrt{\varphi_1(G)} \leq \|C_n(G)_k\|_2 \leq \sqrt{G_0^2 + |k|^2\left(-G_0^2 + \varphi_1(G)\right)}\sqrt{1 - G_0^2 + \varphi_1(G)},$$

and if $|k| < 1$, *then*

$$|k|\sqrt{\varphi_1(G)} \leq \|C_n(G)_k\|_2 \leq \sqrt{n\left(\varphi_1(G)\right)},$$

where $\varphi_1(G)$ *is as in Corollary 2.10.*

Proof. Take $V_n = G_n,\ G_0 = 1, G_1 = 0, G_2 = 2$ in Theorem 2.13. □

Fourthly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(B)_k$.

Corollary 2.17. *Let $C_n(B)_k = \mathrm{Circ}_k(B_0, B_1, \dots, B_{n-1})$ be a third-order Lucas-Pell k-circulant matrix. Then if $|k| \geq 1$, then*

$$\sqrt{\varphi_1(B)} \leq \|C_n(B)_k\|_2 \leq \sqrt{B_0^2 + |k|^2\left(-B_0^2 + \varphi_1(B)\right)}\sqrt{1 - B_0^2 + \varphi_1(B)},$$

and if $|k| < 1$, then

$$|k|\sqrt{\varphi_1(B)} \leq \|C_n(B)_k\|_2 \leq \sqrt{n\left(\varphi_1(B)\right)},$$

where $\varphi_1(B)$ is as in Corollary 2.10.

Proof. Take $V_n = B_n,\ B_0 = 3, B_1 = 0, B_2 = 4$ in Theorem 2.13. □

Here, we present the determinant of $C_n(V)_k$.

Theorem 2.18. *The determinant of $C_n(V)_k$ is given by*

$$\det(C_n(V)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1)},$$

where

$$\begin{aligned}
\Lambda_1 &= kV_n - V_0, \\
\Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1} + V_1), \\
\Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2} - 2kV_n - V_2 + 2V_0).
\end{aligned}$$

Proof. By considering the following identities

$$\begin{aligned}
\prod_{k=0}^{n-1}(x - y\omega^{-k}) &= x^n - y^n, \\
\prod_{j=0}^{n-1}(x - y\omega^{-j} + z\omega^{-2j}) &= x^n\left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right),
\end{aligned}$$

and

$$(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1 = (\alpha k^{\frac{1}{n}}\omega^{-j} - 1)(\beta k^{\frac{1}{n}}\omega^{-j} - 1)(\gamma k^{\frac{1}{n}}\omega^{-j} - 1),$$

we have that

$$\prod_{j=0}^{n-1}\left((k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1\right) = (-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1),$$

and

$$\prod_{j=0}^{n-1}\Phi_j(V) = \Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right).$$

where

$$\begin{aligned}
\omega &= \exp(2\pi i/n), \\
\Phi_j(V) &= kV_n - V_0 - k^{\frac{1}{n}}(-kV_{n+1} + V_1)\omega^{-j} + k^{\frac{2}{n}}(kV_{n+2} - 2kV_n - V_2 + 2V_0)\omega^{-2j},
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_1 &= kV_n - V_0, \\
\Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1} + V_1), \\
\Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2} - 2kV_n - V_2 + 2V_0).
\end{aligned}$$

From Theorem 2.11, we obtain

$$\begin{aligned}
\det(C_n(V)_k) &= \prod_{j=0}^{n-1} \lambda_j(C_n(V)_k) \\
&= \prod_{j=0}^{n-1} \frac{\Phi_j(V)}{(k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1} \\
&= \frac{\prod\limits_{j=0}^{n-1} \Phi_j(V)}{\prod\limits_{j=0}^{n-1}\left((k^{\frac{1}{n}}\omega^{-j})^3 + 2(k^{\frac{1}{n}}\omega^{-j})^2 - 1\right)} \\
&= \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1)}.
\end{aligned}$$

This completes the proof. □

We consider four special cases of Theorem 2.18.

First, the following corollary gives the determinant of $C_n(R)_k$.

Corollary 2.19. *The determinant of $C_n(R)_k$ is given by*

$$\det(C_n(R)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1)},$$

where

$$\begin{aligned}
\Lambda_1 &= kR_n - 1, \\
\Lambda_2 &= k^{\frac{1}{n}}(-kR_{n+1} + 1), \\
\Lambda_3 &= k^{\frac{2}{n}}(kR_{n+2} - 2kR_n + 1).
\end{aligned}$$

Proof. Take $V_n = R_n,\ R_0 = 1, R_1 = 1, R_2 = 1$ in Theorem 2.18. □

Second, the following corollary presents the determinant of $C_n(C)_k$.

Corollary 2.20. *The determinant of $C_n(C)_k$ is given by*

$$\det(C_n(C)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1)},$$

where

$$\begin{aligned}
\Lambda_1 &= kC_n - 3, \\
\Lambda_2 &= k^{\frac{1}{n}}(-kC_{n+1}), \\
\Lambda_3 &= k^{\frac{2}{n}}(kC_{n+2} - 2kC_n + 4).
\end{aligned}$$

Proof. This follows from Theorem 2.18 by substituting $V_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$. □

Third, the following corollary gives the determinant of $C_n(G)_k$.

Corollary 2.21. *The determinant of $C_n(G)_k$ is given by*

$$\det(C_n(G)_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1)},$$

where

$$\begin{aligned} \Lambda_1 &= kG_n - 1, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kG_{n+1}), \\ \Lambda_3 &= k^{\frac{2}{n}}(kG_{n+2} - 2kG_n). \end{aligned}$$

Proof. This follows from Theorem 2.18 by substituting $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$. □

Fourth, the following corollary presents the determinant of $C_n(B)_k$.

Corollary 2.22. *The determinant of $C_n(B)_k$ is given by*

$$\det(C_n(B)_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kB_n + (k - B_{-n})k^2 - 1)},$$

where

$$\begin{aligned} \Lambda_1 &= kB_n - 3, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kB_{n+1}), \\ \Lambda_3 &= k^{\frac{2}{n}}(kB_{n+2} - 2kB_n + 2). \end{aligned}$$

Proof. This follows from Theorem 2.18 by substituting $V_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$. □

3 Conclusion

In this paper, we derived explicit formulas for the eigenvalues, determinant, and various norms of k-circulant matrices involving generalized Pell-Padovan numbers. We also established bounds for the spectral norm. By specializing the initial values, we provided several corollaries for Pell-Padovan, Pell-Perrin, and third-order Fibonacci-Pell sequences, unifying and extending existing results in the literature. These findings offer theoretical foundations and computational tools for applications in fields such as cryptography and signal processing. Future research may extend this methodology to other structured matrices, such as Toeplitz or k-semicirculant matrices.

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Conflict of interest

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