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Numerical solution of some class of differential equations by Galerkin method utilizing Boubaker wavelets

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Abstract. This paper proposes a Galerkin method based on Boubaker wavelets (BWGM) for the numerical solution of a class of differential equations. The method employs Boubaker wavelets as both weight functions and basis elements to construct approximate solutions. The accuracy of the proposed method is evaluated by comparing numerical results with exact solutions and with existing schemes such as the Galerkin method using Fibonacci and Gegenbauer wavelets. Several examples are provided to demonstrate the validity and applicability of the method. The results indicate that BWGM yields high accuracy with minimal absolute error, making it an efficient tool for solving linear, singular, and nonlinear boundary value problems.

Keywords: Boubaker wavelets; function approximation; Galerkin method; differential equations.

2020 Mathematics Subject Classification: 34A40, 26C10, 65T60.

1 Introduction

In recent years, studies of boundary value problems for second-order ordinary differential equations have attracted the attention of many mathematicians and physicists. Moreover, most differential equations arising from the modeling of physical phenomena do not always have known analytical solutions. Thus, the development of numerical approaches to find approximate solutions becomes essential.

Several numerical methods have recently been used for the numerical solution of ordinary differential equations, such as the Haar wavelet collocation method [14], the Legendre wavelet collocation method [12], and the Laguerre wavelet-Galerkin method [13], among others.

Wavelets have become a popular topic in many scientific and engineering discussions. They are recognized as a new basis for representing functions, a technique for timefrequency analysis, and a significant mathematical subject. Wavelet analysis is a numerical concept that

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allows one to represent a function in terms of a set of basis functions called wavelets, which are localized in both location and scale [2].

In wavelet theory, the contributions of Daubechies [7] on orthogonal bases of compactly supported wavelets and Beylkin et al. [5] on the fast wavelet transform algorithm have made wavelet-based approximation of ordinary differential equations particularly attractive. Special interest has been dedicated to the construction of compactly supported smooth wavelet bases. Spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization but poor spectral localization. Wavelet bases succeed in combining the advantages of both approaches. One strategy for studying differential equations is to use wavelet function bases in place of conventional piecewise polynomial trial functions in finite element-type methods. The Galerkin method is widely used in applied mathematics due to its simplicity and ease of implementation [1,10].

The wavelet-Galerkin method offers advantages over finite difference or finite element methods and has led to numerous applications in science and engineering. To a certain extent, wavelet techniques provide strong competition to the finite element method. Wavelet methods offer an efficient alternative for solving differential equations numerically, especially boundary value problems.

Boubaker wavelets are orthonormal functions, a property that greatly simplifies calculations and leads to straightforward computation of expansion coefficients when used in numerical methods such as collocation or Galerkin methods.

In this paper, we develop a Galerkin method utilizing Boubaker wavelets (BWGM) for the numerical solution of a class of differential equations. The method is based on expanding the solution in terms of Boubaker wavelets with unknown coefficients. The properties of Boubaker wavelets, together with the Galerkin method, are used to evaluate these unknown coefficients and thereby obtain a numerical solution.

The paper is organized as follows. Section 2 introduces Boubaker wavelets and function approximation. Section 3 describes the Boubaker wavelet-based Galerkin method for solving boundary value problems. Numerical implementation is presented in Section 4. Finally, conclusions are discussed in Section 5.

2 Boubaker wavelets and function approximation

2.1 Boubaker wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and translation parameter b vary continuously, we obtain the following family of continuous wavelets [11,15]:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right), \quad \forall a, b \in \mathbb{R}, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-n}$, $b = mb_0a_0^{-n}$, $a_0 > 1$, $b_0 > 0$, we obtain the following family of discrete wavelets:

$$\psi_{n,m}(t) = |a|^{\frac{-1}{2}} \psi(a_0^n t - mb_0), \quad \forall n, m \in Z.$$

The functions $\psi_{n,m}$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, $\psi_{n,m}(t)$ forms an orthonormal basis.

Boubaker wavelets are defined as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2m+1}\dfrac{2m!}{m!^2}B_m(2^{k+1}t-2n+1), & \dfrac{n-1}{2^{k-1}} \le t \le \dfrac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

where k is a positive integer that defines the scale or resolution level of the wavelet basis, controlling how the signal is decomposed; $n = 1,2,3,\dots,2^{k-1}$ is an argument; and $m = 0,1,2,3,\dots,M-1$ is the order of the Boubaker functions.

The orthogonal Boubaker polynomials of m^{th} degree are defined on the interval $[0,1)$ as

$$B_m(t) = \frac{(m!)^2}{(2m)!}\sum_{k=0}^{m}(-1)^{m+k}\frac{(m+k)!}{(m-k)!(k!)^2}t^k.$$

The first few Boubaker polynomials are given below:

$$B_0(t) = 1,$$
$$B_1(t) = \tfrac{1}{2}(2t-1), \tag{2.2}$$
$$B_2(t) = \tfrac{1}{6}(6t^2-6t+1), \text{ and so on.} \tag{2.3}$$

For instance, for $k = 1$ and $M = 3$, we obtain the Boubaker wavelet bases as follows:

$$\psi_{1,0}(t) = 2,$$
$$\psi_{1,1}(t) = 2\sqrt{3}(8t-3), \tag{2.4}$$
$$\psi_{1,2}(t) = 2\sqrt{5}(96t^2-72t+13), \text{ and so on.} \tag{2.5}$$

Function approximation.

Suppose $y(t) \in L^2[0,1)$ is expanded in terms of Boubaker wavelets as

$$y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t). \tag{2.6}$$

Truncating the above infinite series, we get

$$y(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t). \tag{2.7}$$

Convergence of Boubaker wavelets.

Theorem 2.1. *If a continuous function $y(t) \in L^2(\mathbb{R})$ defined on $[0,1)$ is bounded, i.e., $|y(t)| \le K$, then the Boubaker wavelets expansion of $y(t)$ converges uniformly to it [15].*

3 Method of solution

Consider the following boundary value problem:

$$\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = \phi(t), \tag{3.1}$$

with boundary conditions

$$y(0) = a, \quad y(1) = b, \tag{3.2}$$

where $P(t)$ and $Q(t)$ are constants or functions of t, and $\phi(t)$ is a continuous function. Write Eq. (3.1) as

$$R(t) = \frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y - \phi(t), \tag{3.3}$$

where $R(t)$ is the residual of Eq. (3.1); $R(t) = 0$ for the exact solution, and $y(t)$ satisfies the boundary conditions. Consider the trial series solution of Eq. (3.1), where $y(t)$ defined over $[0,1)$ can be expanded in modified Boubaker wavelets, satisfying the given boundary conditions, as follows:

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t), \tag{3.4}$$

where $c_{n,m}$ are unknown coefficients to be determined. Accuracy in the solution is increased by choosing higher-degree Boubaker wavelet polynomials. Differentiating Eq. (3.4) twice with respect to t and substituting into Eq. (3.3), we obtain the residual. To find $c_{n,m}$, we choose weight functions as the assumed basis elements and enforce the orthogonality of the residual to the basis functions over the domain [6]:

$$\int_0^1 \psi_{n,m}(t)R(t)\,dt = 0, \quad \text{for all } n, m.$$

This yields a system of linear algebraic equations. Solving this system gives the unknown parameters. Substituting these into the trial solution Eq. (3.4) yields the numerical solution of Eq. (3.1). To assess the accuracy of the BWGM for the test problems, we use the maximum absolute error:

$$E_{\max} = \max|y_e(t) - y_a(t)|,$$

where $y_e(t)$ and $y_a(t)$ are the exact and approximate solutions, respectively.

4 Numerical implementation

Problem 4.1. Consider the boundary value problem [9]:

$$\frac{d^2y}{dt^2} - 4y = 4\cosh(1), \quad 0 \le t \le 1, \tag{4.1}$$

with boundary conditions:

$$y(0) = 0, \quad y(1) = 0. \tag{4.2}$$

The implementation as per the method explained in Section 3 is as follows:

$$R(t) = \frac{d^2y}{dt^2} - 4y - 4\cosh(1). \tag{4.3}$$

Now, choose the weight function $w(t) = t(t-1)$ for the Boubaker wavelet bases to satisfy the given boundary conditions Eq. (4.2), i.e., $\tilde{\psi}(t) = w(t) \times \psi(t)$:

$$\tilde{\psi}_{1,0}(t) = \psi_{1,0}(t) \times t(1-t) = 2t(1-t),$$

$$\tilde{\psi}_{1,1}(t) = \psi_{1,1}(t) \times t(1-t) = 2\sqrt{3}(8t-3)t(1-t), \tag{4.4}$$

$$\tilde{\psi}_{1,2}(t) = \psi_{1,2}(t) \times t(1-t) = 2\sqrt{5}(96t^2-72t+13)t(1-t). \tag{4.5}$$

Assume the trial solution of Eq. (4.1) for $k = 1$ and $M = 3$ is given by

$$y(t) = c_{1,0}\tilde{\psi}_{1,0}(t) + c_{1,1}\tilde{\psi}_{1,1}(t) + c_{1,2}\tilde{\psi}_{1,2}(t). \tag{4.6}$$

Then Eq. (4.6) becomes

$$y(t) = c_{1,0}2t(1-t) + c_{1,1}2\sqrt{3}(8t-3)t(1-t) + c_{1,2}2\sqrt{5}(96t^2 - 72t + 13)t(1-t). \tag{4.7}$$

Differentiating Eq. (4.7) twice with respect to t and substituting the values of y, $\frac{d^2y}{dt^2}$ into Eq. (4.3) gives the residual of Eq. (4.1). The weight functions are the same as the basis functions. Then, by the weighted Galerkin method, we consider

$$\int_0^1 \tilde{\psi}_{1,j}(t)R(t)\,dt = 0, \quad j = 0, 1, 2. \tag{4.8}$$

For $j = 0, 1, 2$ in Eq. (4.8):

$$\int_0^1 \tilde{\psi}_{1,0}(t)R(t)\,dt = 0, \tag{4.9}$$

$$\int_0^1 \tilde{\psi}_{1,1}(t)R(t)\,dt = 0, \tag{4.10}$$

$$\int_0^1 \tilde{\psi}_{1,2}(t)R(t)\,dt = 0. \tag{4.11}$$

From Eq. (4.9)-(4.11), we obtain a system of algebraic equations with unknown coefficients $c_{1,0}, c_{1,1}$, and $c_{1,2}$. Solving this system yields $c_{1,0} = -1.0933$, $c_{1,1} = 0.0064$, and $c_{1,2} = -0.0016$. Substituting these values into Eq. (4.7) gives the numerical solution. A comparison of the numerical solution and absolute errors is presented in Table 4.1, and the numerical solution alongside the exact solution $y(t) = \cosh(2t-1) - \cosh(1)$ of Eq. (4.1) is shown in Figure 4.1.

Table 4.1: Comparison of numerical solution and absolute error with exact solution of Problem 4.1.

t	FDM Sol.	BWGM Sol.	Exact Sol.	FDM error	BWGM error
0.1	-0.254627	-0.205537	-0.205646	4.90e-02	1.09e-04
0.2	-0.450077	-0.357616	-0.357612	9.25e-02	4.00e-06
0.3	-0.588305	-0.462040	-0.462008	1.26e-01	3.20e-05
0.4	-0.670693	-0.522964	-0.523014	1.48e-01	5.00e-05
0.5	-0.698064	-0.542896	-0.543081	1.55e-01	1.85e-04
0.6	-0.670693	-0.522694	-0.523014	1.48e-01	3.20e-04
0.7	-0.588305	-0.461567	-0.462008	1.26e-01	4.41e-04
0.8	-0.450077	-0.357075	-0.357612	9.25e-02	5.37e-04
0.9	-0.254627	-0.205132	-0.205646	4.90e-02	5.14e-04

Problem 4.2. Consider the boundary value problem [3]:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + y = t^2 - t^3 - 9t + 4, \quad 0 \le t \le 1, \tag{4.12}$$

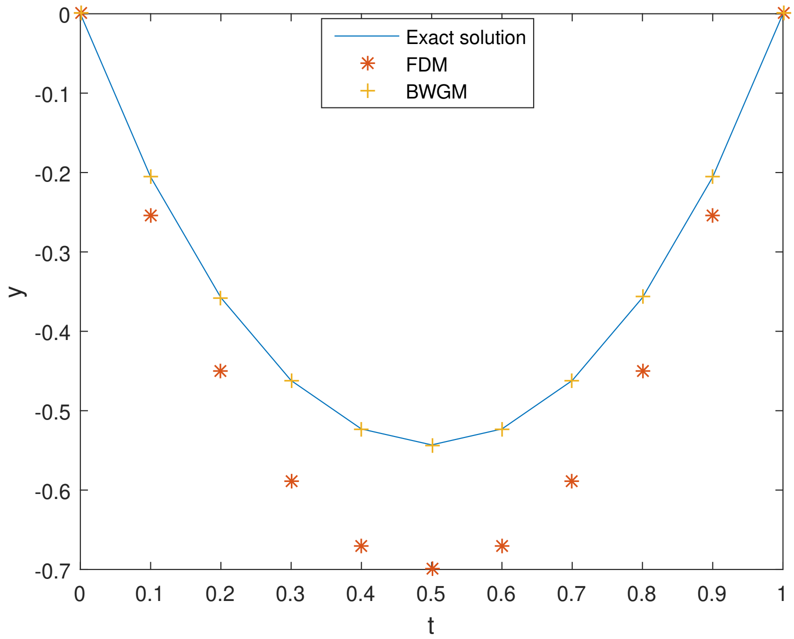

Figure 4.1: Numerical solution vs. exact solution for Problem 4.1.

with boundary conditions:

$$y(0) = 0, \quad y(1) = 0. \tag{4.13}$$

As explained in Section 3 and the previous problem, we obtain $c_{1,0} = 0.18746$, $c_{1,1} = 0.03609$, and $c_{1,2} = 0.00001$. Substituting these values into Eq. (4.7) yields the numerical solution. A comparison of the numerical solution and absolute errors is presented in Table 4.2, and the numerical solution alongside the exact solution $y(t) = t^2 - t^3$ of Eq. (4.12) is shown in Figure 4.2.

Problem 4.3. Consider the boundary value problem [8]:

$$\frac{d^2y}{dt^2} + y^2 = 2\pi^2 \cos(2\pi t) - \sin^4(2\pi t), \quad 0 \leq t \leq 1, \tag{4.14}$$

with boundary conditions:

$$y(0) = 0, \quad y(1) = 0. \tag{4.15}$$

The exact solution of Eq. (4.14) is $y(t) = \sin^2(\pi t)$. The numerical solution, derived as described in Section 3, is compared with the exact solution in Table 4.3 and Figure 4.3.

Table 4.2: Comparison of numerical solution and absolute error with exact solution of Problem 4.2.

t	Ref. [3] Sol.	GWGM Sol.	BWGM Sol.	Exact Sol.	Ref. [3] error	GWGM error	BWGM error
0.1	0.009677	0.0090397	0.008989	0.009000	6.77e-04	3.97e-05	1.10e-05
0.2	0.032675	0.0320510	0.031988	0.032000	6.75e-04	5.10e-05	1.20e-05
0.3	0.063354	0.0630460	0.062993	0.063000	3.54e-04	4.60e-05	7.00e-06
0.4	0.095981	0.0960340	0.096003	0.096000	1.90e-05	3.40e-05	3.00e-06
0.5	0.124731	0.1250217	0.125014	0.125000	2.69e-04	2.17e-05	1.40e-05
0.6	0.143688	0.1440131	0.144023	0.144000	3.12e-04	1.31e-05	2.30e-05
0.7	0.146841	0.1470093	0.147030	0.147000	1.59e-04	9.30e-06	3.00e-05
0.8	0.128089	0.1280091	0.128029	0.128000	8.90e-05	9.10e-06	2.90e-05
0.9	0.080862	0.0810082	0.081020	0.081000	1.38e-04	8.20e-06	2.00e-05

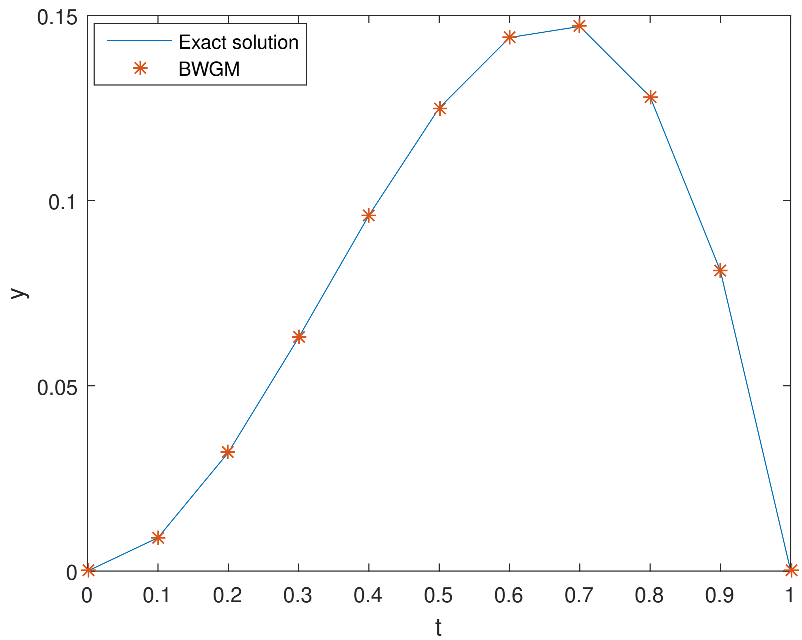

Figure 4.2: Numerical solution vs. exact solution for Problem 4.2.

Table 4.3: Comparison of numerical solution and absolute error with exact solution of Problem 4.3.

t	Ref. [4] Sol.	BWGM Sol.	Exact Sol.	Ref. [4] error	BWGM error
0.1	0.096787	0.096578	0.095492	1.30e-03	1.09e-03
0.2	0.350839	0.350293	0.345492	5.35e-03	4.80e-03
0.3	0.656318	0.656028	0.654508	1.81e-03	1.52e-03
0.4	0.905968	0.905886	0.904508	1.46e-03	1.38e-03
0.5	0.998985	0.998999	1.000000	1.02e-03	1.00e-03
0.6	0.910215	0.910052	0.904508	5.71e-03	5.54e-03
0.7	0.656335	0.655638	0.654508	1.83e-03	1.13e-03
0.8	0.346849	0.346835	0.345492	1.36e-03	1.34e-03
0.9	0.097656	0.097365	0.095492	2.16e-03	1.87e-03

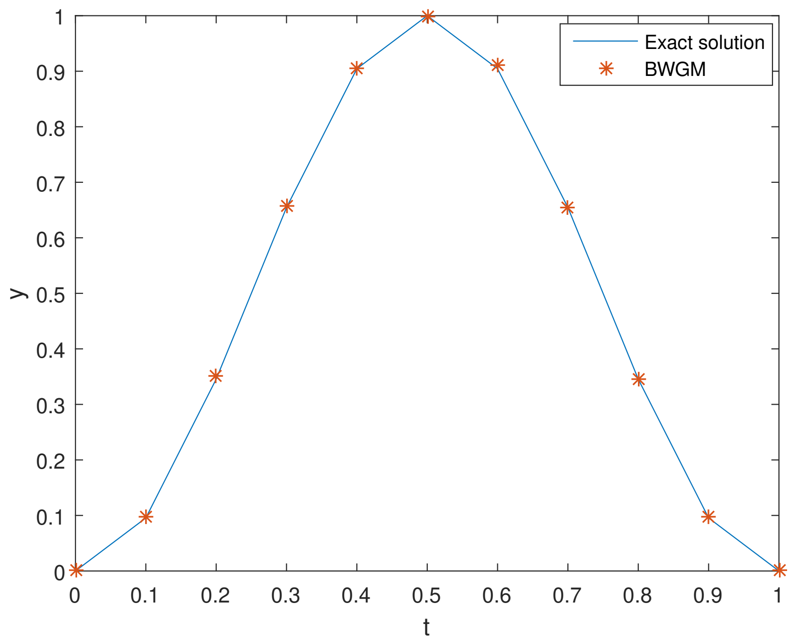

Figure 4.3: Numerical solution vs. exact solution for Problem 4.3.

A comparison of absolute errors for Problems 4.1 to 4.3 is given below.

5 Conclusions

In this paper, a BWGM is proposed for the numerical solution of a class of differential equations. From the tables and figures above, we observe the following:

- The numerical solutions obtained by this method are more accurate than those obtained by the finite difference method (FDM) and other methods such as those using Fibonacci wavelets [3] and Gegenbauer wavelets (GWGM).
- The absolute error from this method is very small compared to the FDM and other methods, such as those detailed in Ref. [3] (Fibonacci wavelets) and GWGM (Gegenbauer wavelets).

This development advances recent research in numerical analysis and provides significant benefits to researchers. Thus, the Galerkin method utilizing Boubaker wavelets is very efficient for linear, singular, and nonlinear boundary value problems.

Declarations

Availability of data and materials

Not applicable.

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Authors' contributions

The author has alone contributed to this paper.

Conflict of interest

The author has no conflicts of interest to declare.

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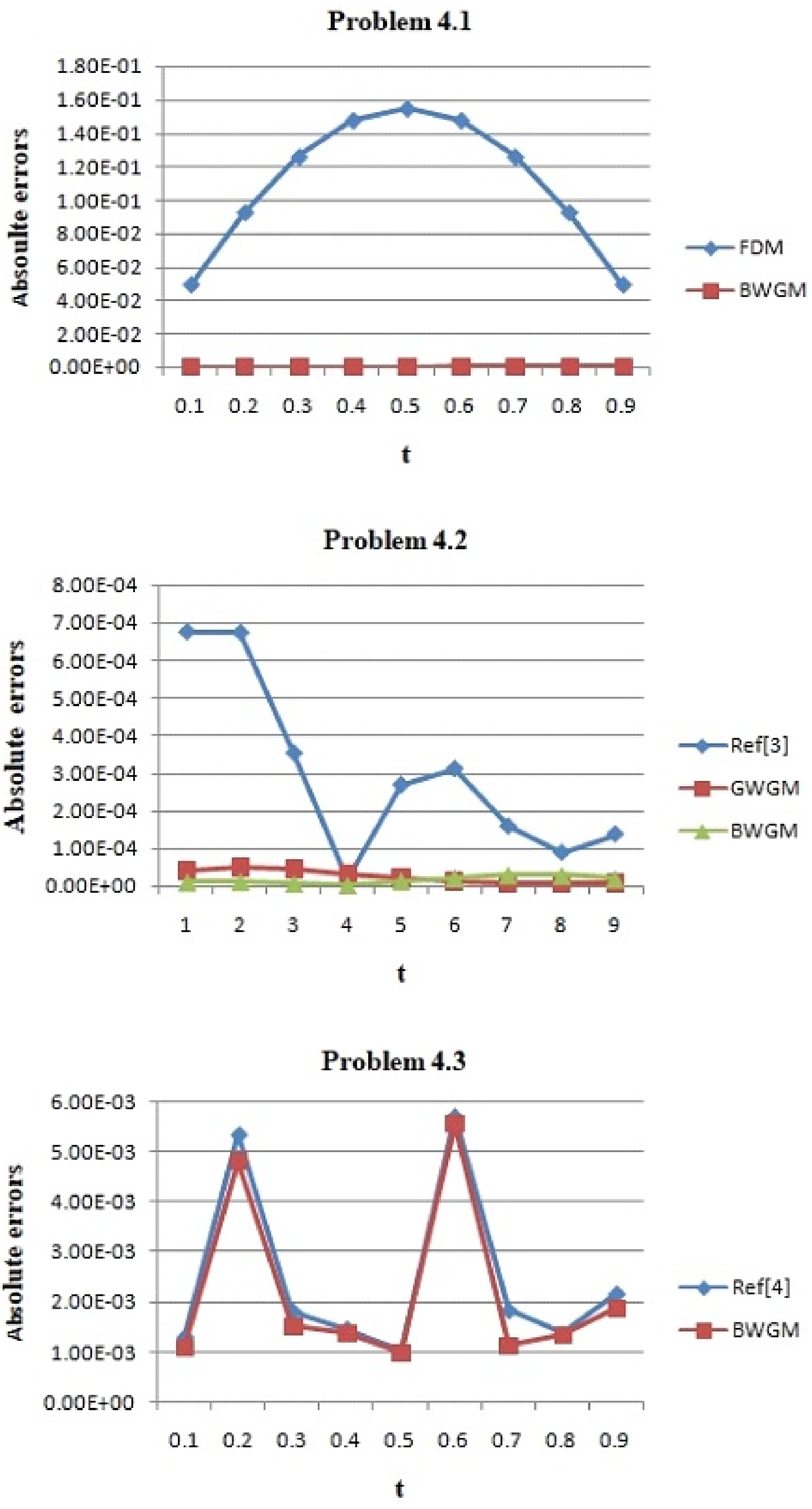

Figure 4.4: Comparison of absolute errors for Problems 4.1–4.3.

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