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Nonlinear impulsive fractional differential equation with Atangana-Baleanu-Caputo derivative

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Abstract. In this work, we study the existence and uniqueness of solutions to a class of fractional differential equations of ABC type with impulse conditions. Sufficient conditions are established to prove the existence results by using fixed point theorems such as the Banach contraction principle and the Schauder fixed point theorem. In addition, the Ulam stability of the solution is considered. Finally, an example is provided to illustrate the main results.

Keywords: Fractional differential equation, Atangana-Baleanu-Caputo derivative, Fixed point.

2020 Mathematics Subject Classification: 26A33, 47H10.

1 Introduction

The history of fractional calculus dates back more than 300 years. The concept of a fractional derivative can be traced to LHospital, who posed the question of the meaning of a derivative of order $1/2$. In recent years, researchers have devoted increasing attention to fractional derivatives, especially following their emergence in diverse applications across biology, economics, science, and engineering [1,3,9,10,13,19]. Owing to their wide range of applications, fractional-order differential equations have become a rich and active area of applied mathematics.

Fractional differential equations (FDEs) with singular kernels present challenges when modeling certain physical phenomena. To overcome these difficulties, researchers have introduced new types of fractional operators that are free of singularities. In 2015, Caputo et al. [10] proposed a definition based on the exponential function. This led to a series of studies that advanced the theory of fractional calculus within the framework of the Caputo -Fabrizio (CF) fractional derivative. For a broader overview, we refer the reader to [2,14] and the references therein.

One of the most prominent nonsingular fractional derivatives is the Atangana -Baleanu (AB) derivative, introduced by Atangana and Baleanu in [4]. The Caputo-type version of this

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operator, known as the ABC derivative, was also introduced in the same work. This new formulation has proven to be both more applicable and more effective in a wide range of problems. Since then, many researchers have contributed to the development of fractional calculus in the context of AB and ABC derivatives. Notable contributions include the works in [18,20].

Impulsive differential equations have played an important role in modeling various phenomena. Since the last century, several authors have used impulsive differential systems to describe models, particularly in the context of population dynamics subject to abrupt changes, as well as phenomena such as harvesting, the spread of diseases, and more; see [7,8].

In [17], Reunsumrit et al. discussed existence results for impulsive fractional integro-differential equations involving the Atangana -Baleanu -Caputo derivative under integral boundary conditions. Wattanakejorn et al. [21] studied implicit fractional relaxation differential equations with impulsive delay boundary conditions.

In [6], Benchohra et al. investigated the existence and stability of solutions for a boundary value problem involving nonlinear implicit fractional differential equations with impulses:

$$\begin{aligned}
&{}^{c}D^{\alpha}_{t_k}y(t) = f\left(t, y, {}^{c}D^{\alpha}_{t_k}y(t)\right), \quad t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \alpha < 1,\\
&\Delta y\big|_{t=t_k} = I_k\left(y(t_k^-)\right), \quad k = 1, \dots, m,\\
&ay(1) + by(t) = c,
\end{aligned}$$

where ${}^{c}D^{\alpha}_{t_k}$ denotes the Caputo -Hadamard fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and a, b, and c are real constants with $a + b \neq 0$.

Yadav and Pandey [22] studied the existence and uniqueness of solutions to fractional differential equations with impulsive boundary conditions and Atangana -Baleanu (AB) fractional integral:

$$\begin{aligned}
&{}^{CH}D^{\lambda}_{\sigma_k}\varphi(\sigma) = f(\sigma, \varphi(\sigma), {}^{AB}I^{\mu}_{i}(\varphi(\sigma))), \quad \sigma, \sigma_k \in [t_0, T],\ \sigma \neq \sigma_k,\\
&\Delta\varphi(\sigma_k) = I_k(\varphi(\sigma_k)), \quad k = 1, \dots, m,\\
&\Delta\delta\varphi(\sigma_k) = I_k(\varphi(\sigma_k)), \quad k = 1, \dots, m,\\
&\varphi(t_0) = h(\varphi), \quad \varphi(T) = g(\varphi),
\end{aligned}$$

where ${}^{CH}D^{\lambda}_{\sigma_k}$ is the Caputo -Hadamard fractional derivative, ${}^{AB}I^{\mu}_{i}$ is the Atangana -Baleanu fractional integral, and $f : [t_0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Motivated by the above results, we consider the Atangana -Baleanu fractional derivative with impulsive boundary conditions:

$$\begin{cases}
{}^{ABC}D^{\alpha}_{t}\vartheta(t) = \phi(t, \vartheta, {}^{ABC}D^{\alpha}_{t}\vartheta(t)), & t \in [0, T] = J,\ 0 < \alpha < 1,\\
\Delta\vartheta(t_k) = \vartheta(t_k^+) - \vartheta(t_k^-) = I_k(\vartheta(t_k^-)), & k = 1, \dots, m,\\
\vartheta(0) = \vartheta_0,
\end{cases} \tag{1.1}$$

where ${}^{ABC}D^{\alpha}_{t}$ is the ABC fractional derivative of order α, $\phi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $I_k : \mathbb{R} \to \mathbb{R}$. Here $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq t_{m+1} = T$, $\Delta\vartheta(t_k) = I_k(\vartheta(t_k^-)) = \vartheta(t_k^+) - \vartheta(t_k^-)$, $\vartheta(t_k^+) = \lim_{h\to 0^+} \vartheta(t_k + h)$ and $\vartheta(t_k^-) = \lim_{h\to 0^-} \vartheta(t_k - h)$ represent the right and left limits of $\vartheta(t)$ at $t = t_k$.

This manuscript is organized as follows. In Section 2, some definitions and lemmas are given which are used in the main results. In Section 3, the existence of a solution for the nonlinear fractional differential equation is obtained by Banach and Schauder fixed point theorems. Section 4 discusses the Ulam stability of solutions. Finally, an example is given to support our main theoretical result.

2 Preliminaries

Definition 2.1. Let $p \in [1, \infty)$ and Ω be an open subset of $\mathbb{R}$. The Sobolev space $H^p(\Omega)$ is defined by

$$H^p(\Omega) = \{f \in L^p(\Omega) : D^\beta f \in L^p(\Omega), \text{ for all } |\beta| \leq p\}.$$

Definition 2.2. Let $f \in H^1(0,1)$ and $0 < \alpha < 1$. The Atangana -Baleanu fractional derivative of Caputo sense is defined by

$$^{ABC}D^\alpha f(x) = \frac{\beta(\alpha)}{1-\alpha} \int_0^x f'(s)\, \mathcal{E}_\alpha\Big[-\frac{\alpha}{1-\alpha}(x-s)^\alpha\Big] ds,$$

where $\beta(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}$ is the normalization function which meets $\beta(0) = \beta(1) = 1$, and $\mathcal{E}_\alpha$ is the well-known Mittag-Leffler function of one variable.

Definition 2.3. Let $f \in H^1(0,1)$ and $0 < \alpha < 1$. The Atangana -Baleanu fractional derivative of Riemann-Liouville sense is defined by

$$^{ABR}D^\alpha f(x) = \frac{\beta(\alpha)}{1-\alpha} \frac{d}{dx} \int_0^x f(s)\, \mathcal{E}_\alpha\Big[-\frac{\alpha}{1-\alpha}(x-s)^\alpha\Big] ds.$$

The associated fractional integral is defined by

$$^{AB}I^\alpha f(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^x f(s)\,(x-s)^{\alpha-1} ds.$$

We refer the readers to [2,4,15,16] for more details about the definition and properties of the derivative.

Lemma 2.4. *Consider the problem*

$$^{ABC}D^\alpha x(t) = h(t), \qquad x(0) = x_0.$$

Then the solution is given by

$$x(t) = x_0 + \frac{1-\alpha}{\beta(\alpha)} h(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

Theorem 2.5 ([23]; Ascoli -Arzela theorem)**.** *Let $A \subset PC(J, \mathbb{R})$. Then A is relatively compact if:*

1. *A is uniformly bounded, i.e., there exists $M > 0$ such that $|f(x)| < M$ for every $f \in A$ and $x \in (t_k, t_{k+1})$, $k = 1, \ldots, m$;*
2. *A is equicontinuous on (t_k, t_{k+1}), i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_1, x_2 \in (t_k, t_{k+1})$ with $|x_1 - x_2| \leq \delta$ implies $|f(x_1) - f(x_2)| \leq \varepsilon$ for every $f \in A$.*

Theorem 2.6 ([5]; Banach's fixed point theorem)**.** *Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T from C into itself has a unique fixed point.*

Theorem 2.7 ([5]; Schauder fixed point theorem)**.** *If K is a closed bounded convex subset of a Banach space X and $W : K \to K$ is completely continuous, then W has at least one fixed point in K.*

Lemma 2.8. *[11,12] Let for $t \geq t_0 \geq 0$ the following inequality hold:*

$$x(t) \leq a(t) + \int_{t_0}^t b(t,s)x(s)ds + \sum_{t_0 < t_k < t} \beta_k x(t_k),$$

where $\beta_k \geq 0$ are constants, $a \in PC([t_0, \infty), \mathbb{R}^+)$ is nondecreasing, and $b(t,s)$ is a continuous nonnegative function for $t, s \geq t_0$ and is nondecreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$,

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k) \exp \int_{t_0}^t b(t,s)ds.$$

3 Main results

Let $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$. Consider the following space

$$\begin{aligned} PC(J, \mathbb{R}) = \{ & y\colon J \to \mathbb{R},\ y \in C((t_k, t_{k+1}], \mathbb{R}) \text{ for } k = 0, \ldots m, \\ & \text{and there exist } \ y(t_k^-) \ \text{ and } \ y(t_k^+) \ \text{ with } \ y(t_k) = y(t_k^-), \ \ k = 1, \cdots, m\}, \end{aligned}$$

Clearly, $PC(J, \mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} |y(t)|.$$

Lemma 3.1. *Consider the nonlinear integral boundary value problem*

$$\begin{cases} {}^{ABC}D_t^{\alpha} \vartheta(t) = f(t), & t \in [0, T] = J,\ 0 < \alpha < 1, \\ \Delta\vartheta(t_k) = \vartheta(t_k^+) - \vartheta(t_k^-) = I_k(\vartheta(t_k^-)), & k = 1, \ldots, m, \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{3.1}$$

Then the solution of (3.1) *is*

$$\vartheta(t) = \begin{cases} \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, & t \in [0, t_1], \\ \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t) + \sum\limits_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)} f(t_i) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum\limits_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s)\, ds & \\ + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} f(s)\, ds + \sum\limits_{i=1}^{m} I_i(\vartheta(t_i^-)), & t \in [t_k, t_{k+1}]. \end{cases} \tag{3.2}$$

Proof. Assume ϑ satisfies (3.1). If $t \in [0, t_1]$, then ${}^{ABC}D_t^{\alpha} \vartheta(t) = f(t)$. Lemma 2.4 implies

$$\vartheta(t) = \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} \vartheta(t) &= \vartheta(t_1^+) + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s)\, ds \\ &= \Delta(\vartheta(t_1)) + \vartheta(t_1^-) + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s)\, ds \\ &= I_1(\vartheta(t_1^-)) + \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t_1) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s)\, ds \\ &\quad + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s)\, ds \\ &= \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t_1) + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s)\, ds \\ &\quad + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s)\, ds + I_1(\vartheta(t_1^-)). \end{aligned}$$

If $t \in (t_2, t_3]$, we similarly obtain

$$\begin{aligned}
\vartheta(t) &= \vartheta(t_2^+) + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} f(s)\, ds \\
&= \Delta(\vartheta(t_2)) + \vartheta(t_2^-) + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} f(s)\, ds \\
&= I_2(\vartheta(t_2^-)) + \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t_1) + \frac{1-\alpha}{\beta(\alpha)} f(t_2) \\
&\quad + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} f(s)\, ds + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)\, ds \\
&\quad + I_1(\vartheta(t_1^-)) + \frac{1-\alpha}{\beta(\alpha)} f(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} f(s)\, ds \\
&= \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} f(t_1) + \frac{1-\alpha}{\beta(\alpha)} f(t_2) + \frac{1-\alpha}{\beta(\alpha)} f(t) \\
&\quad + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} f(s)\, ds + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)\, ds \\
&\quad + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} f(s)\, ds + I_1(\vartheta(t_1^-)) + I_2(\vartheta(t_2^-)).
\end{aligned}$$

Repeating this process, for $t \in [t_k, t_{k+1}]$, $k = 1, \dots, m$, we obtain

$$\begin{aligned}
\vartheta(t) = \vartheta_0 &+ \frac{1-\alpha}{\beta(\alpha)} f(t) + \sum_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)} f(t_i) \\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} f(s)\, ds \\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} f(s)\, ds + \sum_{i=1}^{m} I_i(\vartheta(t_i^-)).
\end{aligned}$$

The converse is obtained by direct differentiation. This completes the proof. □

Lemma 3.2. *A function $\vartheta \in PC(J, \mathbb{R})$ is a solution of* (1.1) *if and only if*

$$\vartheta(t) = \begin{cases} \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} \psi(t) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s)\, ds, & t \in [0, t_1], \\ \vartheta_0 + \frac{1-\alpha}{\beta(\alpha)} \psi(t) + \sum\limits_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)} \psi(t_i) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum\limits_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} \psi(s)\, ds & \\ + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} \psi(s)\, ds + \sum\limits_{i=1}^{m} I_i(\vartheta(t_i^-)), & t \in [t_k, t_{k+1}], \end{cases} \tag{3.3}$$

where $\psi(t) = \phi(t, \vartheta(t), \psi(t))$.

The following assumptions are needed to prove our first existence result for (1.1) by using Banach's fixed point theorem.

(H1) There exist constants $\lambda_1 > 0$ and $0 < \lambda_2 < 1$ such that for any $u_1, u_2, v_1, v_2 \in \mathbb{R}$,

$$|\phi(t, u_1, u_2) - \phi(t, v_1, v_2)| \le \lambda_1 |u_1 - v_1| + \lambda_2 |u_2 - v_2|.$$

(H2) There exists a constant $\kappa > 0$ such that for any $u, v \in \mathbb{R}$,

$$|I_k(u) - I_k(v)| \le \kappa |u - v|.$$

Theorem 3.3. *Assume that hypotheses (H1) and (H2) hold. Then the problem* (1.1) *has a unique solution if*

$$\Big[\Big(\frac{\lambda_1(1-\alpha)}{\beta(\alpha)(1-\lambda_2)}+\frac{\lambda_1 T^\alpha}{\beta(\alpha)\Gamma(\alpha)(1-\lambda_2)}\Big)(m+1)+m\kappa\Big]<1.$$

Proof. Consider the operator $\mathcal{N}: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ defined by

$$\begin{aligned}\mathcal{N}(\vartheta)(t)&=\vartheta_0+\frac{1-\alpha}{\beta(\alpha)}\psi(t)+\sum_{i=1}^m\frac{1-\alpha}{\beta(\alpha)}\psi(t_i)\\&\quad+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\psi(s)\,ds\\&\quad+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^t(t-s)^{\alpha-1}\psi(s)\,ds+\sum_{i=1}^m I_i(\vartheta(t_i^-)),\end{aligned}$$

where $\psi\in C(J,\mathbb{R})$ is given by $\psi(t)=\phi(t,\vartheta(t),\psi(t))$.

We shall prove that $\mathcal{N}$ is a contraction. Let $\vartheta,\tilde{\vartheta}\in PC(J,\mathbb{R})$. Then for $t\in J$,

$$\begin{aligned}&\Big|\mathcal{N}(\vartheta)(t)-\mathcal{N}(\tilde{\vartheta})(t)\Big|\\&\le\frac{1-\alpha}{\beta(\alpha)}\Big|\psi(t)-\tilde{\psi}(t)\Big|+\sum_{i=1}^m\frac{1-\alpha}{\beta(\alpha)}\Big|\psi(t_i)-\tilde{\psi}(t_i)\Big|\\&\quad+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\Big|\psi(s)-\tilde{\psi}(s)\Big|\,ds\\&\quad+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^t(t-s)^{\alpha-1}\Big|\psi(s)-\tilde{\psi}(s)\Big|\,ds+\sum_{i=1}^m\Big|I_i(\vartheta(t_i^-))-I_i(\tilde{\vartheta}(t_i^-))\Big|,\end{aligned}$$

where $\psi,\tilde{\psi}\in C(J,\mathbb{R})$ are such that

$$\psi(t)=\phi(t,\vartheta(t),\psi(t)),\qquad\tilde{\psi}(t)=\phi(t,\tilde{\vartheta}(t),\tilde{\psi}(t)).$$

For each $t\in J$, we have

$$\begin{aligned}\Big|\psi(t)-\tilde{\psi}(t)\Big|&=\Big|\phi(t,\vartheta(t),\psi(t))-\phi(t,\tilde{\vartheta}(t),\tilde{\psi}(t))\Big|\\&\le\lambda_1\Big|\vartheta(t)-\tilde{\vartheta}(t)\Big|+\lambda_2\Big|\psi(t)-\tilde{\psi}(t)\Big|.\end{aligned}$$

Thus

$$\Big|\psi(t)-\tilde{\psi}(t)\Big|\le\frac{\lambda_1}{1-\lambda_2}\Big|\vartheta(t)-\tilde{\vartheta}(t)\Big|.$$

Consequently,

$$\begin{aligned}\Big|\mathcal{N}(\vartheta)(t)-\mathcal{N}(\tilde{\vartheta})(t)\Big|&\le\frac{\lambda_1(1-\alpha)}{\beta(\alpha)(1-\lambda_2)}\|\vartheta-\tilde{\vartheta}\|+\frac{m\lambda_1(1-\alpha)}{\beta(\alpha)(1-\lambda_2)}\|\vartheta-\tilde{\vartheta}\|\\&\quad+\frac{m\lambda_1T^\alpha}{\beta(\alpha)\Gamma(\alpha)(1-\lambda_2)}\|\vartheta-\tilde{\vartheta}\|+\frac{\lambda_1T^\alpha}{\beta(\alpha)\Gamma(\alpha)(1-\lambda_2)}\|\vartheta-\tilde{\vartheta}\|+m\kappa\|\vartheta-\tilde{\vartheta}\|\\&\le\Big\{\Big(\frac{\lambda_1(1-\alpha)}{\beta(\alpha)(1-\lambda_2)}+\frac{\lambda_1T^\alpha}{\beta(\alpha)\Gamma(\alpha)(1-\lambda_2)}\Big)(m+1)+m\kappa\Big\}\|\vartheta-\tilde{\vartheta}\|.\end{aligned}$$

By the contractivity condition, $\mathcal{N}$ is a contraction and hence has a unique fixed point, which is the unique solution of (1.1). □

Our second result is based on Schauder's fixed point theorem.

Theorem 3.4. *In addition to (H1) and (H2), assume that:*

(H3) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $\iota_1, \iota_2 > 0$ such that

$$|I_k(u)| \leq \iota_1 |u| + \iota_2, \quad \text{for each } u \in \mathbb{R},\ k = 1, \dots, m.$$

(H4) There exists $M^* > 0$ such that $M^* = \sup_{t \in J} |\phi(t, 0, 0)|$.

Then the BVP (1.1) has at least one solution on J.

Proof. Define the closed ball of the Banach space $PC(J, \mathbb{R})$ as

$$\mathcal{B} = \{\vartheta \in PC(J, \mathbb{R}),\ \|\vartheta\|_{PC} \leq R\}$$

for some $R > 0$ to be chosen later. Clearly $\mathcal{B}$ is nonempty, bounded, closed, and convex.

Step 1: $\mathcal{N}$ is continuous.
Let $\vartheta_n \to \vartheta$ in $\mathcal{B}$. Then

$$\begin{aligned}
|\mathcal{N}(\vartheta_n)(t) - \mathcal{N}(\vartheta)(t)| \leq{}& \frac{1-\alpha}{\beta(\alpha)} |\psi_n(t) - \psi(t)| + \sum_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)} |\psi_n(t_i) - \psi(t_i)| \\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\psi_n(s) - \psi(s)|\, ds \\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t - s)^{\alpha-1} |\psi_n(s) - \psi(s)|\, ds \\
&+ \sum_{i=1}^{m} |I_i(\vartheta_n(t_i^-)) - I_i(\vartheta(t_i^-))|,
\end{aligned}$$

where $\psi_n(t) = \phi(t, \vartheta_n(t), \psi_n(t))$ and $\psi(t) = \phi(t, \vartheta(t), \psi(t))$. By continuity of ϕ and I_k, and using Lebesgue dominated convergence, we get $\|\mathcal{N}(\vartheta_n) - \mathcal{N}(\vartheta)\|_{PC} \to 0$. Thus $\mathcal{N}$ is continuous.

Step 2: $\mathcal{N}(\mathcal{B})$ is bounded.
For $\vartheta \in \mathcal{B}$ and $t \in J$, we have

$$\begin{aligned}
|\mathcal{N}(\vartheta)(t)| \leq{}& |\vartheta_0| + \frac{1-\alpha}{\beta(\alpha)} |\psi(t)| + \sum_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)} |\psi(t_i)| \\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\psi(s)|\, ds \\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t - s)^{\alpha-1} |\psi(s)|\, ds + \sum_{i=1}^{m} |I_i(\vartheta(t_i^-))|,
\end{aligned}$$

where $\psi(t) = \phi(t, \vartheta(t), \psi(t))$. From (H1) and (H4),

$$|\psi(t)| = |\phi(t, \vartheta(t), \psi(t))| \leq \lambda_1 |\vartheta(t)| + \lambda_2 |\psi(t)| + M^* \leq R\lambda_1 + \lambda_2 |\psi(t)| + M^*,$$

so

$$\|\psi\| \leq \frac{R\lambda_1 + M^*}{1 - \lambda_2} =: \eta.$$

Thus

$$
\begin{aligned}
|\mathcal{N}(\vartheta)(t)| &\le |\vartheta_0| + \frac{1-\alpha}{\beta(\alpha)}\eta + m\frac{1-\alpha}{\beta(\alpha)}\eta + \frac{mT^\alpha}{\beta(\alpha)\Gamma(\alpha)}\eta + \frac{T^\alpha}{\beta(\alpha)\Gamma(\alpha)}\eta + m(\iota_1 R + \iota_2)\\
&\le |\vartheta_0| + \eta\Big[\frac{1-\alpha}{\beta(\alpha)} + \frac{T^\alpha}{\beta(\alpha)\Gamma(\alpha)}\Big](m+1) + m(\iota_1 R + \iota_2) =: \ell.
\end{aligned}
$$

Choose $R \ge \ell$. Then $\mathcal{N}(\mathcal{B}) \subset \mathcal{B}$.

Step 3: $\mathcal{N}(\mathcal{B})$ is equicontinuous.
Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $\vartheta \in \mathcal{B}$. Then

$$
\begin{aligned}
|\mathcal{N}(\vartheta)(\tau_2) - \mathcal{N}(\vartheta)(\tau_1)| \le{}& \frac{1-\alpha}{\beta(\alpha)}|\psi(\tau_2) - \psi(\tau_1)|\\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^{\tau_1}\big((\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\big)|\psi(s)|\,ds\\
&+ \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}|\psi(s)|\,ds\\
&+ \sum_{t_m\in(\tau_1,\tau_2)} |I_m(\vartheta(t_m^-))|.
\end{aligned}
$$

Since ψ is continuous and $\alpha \in (0,1)$, the right-hand side tends to zero as $\tau_2 \to \tau_1$. Hence $\mathcal{N}(\mathcal{B})$ is equicontinuous.

By the Ascoli -Arzela theorem, $\mathcal{N}$ is completely continuous. Schauder's fixed point theorem then guarantees at least one fixed point, which is a solution of (1.1). □

4 Ulam stability analysis

In this section, we study the Ulam stability of (1.1). Let $\varepsilon > 0$ and consider the inequalities

$$
\begin{cases}
\|{}^{ABC}D_t^\alpha \mu(t) - \phi(t, \mu, {}^{ABC}D_t^\alpha \mu(t))\| \le \varepsilon, & t \in J,\ 0 < \alpha < 1,\\
\|\Delta\mu(t_k) - I_k(\mu(t_k^-))\| \le \varepsilon, & k = 1, \dots, m.
\end{cases}
\tag{4.1}
$$

Definition 4.1. The problem (1.1) is Ulam -Hyers stable if there exists a real number $\rho > 0$ such that, for each $\varepsilon > 0$ and each solution $\mu \in PC^1(J, \mathbb{R})$ of inequality (4.1), there exists a solution $\vartheta \in PC^1(J, \mathbb{R})$ of (1.1) with

$$
|\mu(t) - \vartheta(t)| \le \rho\,\varepsilon, \quad t \in J.
$$

Definition 4.2. The problem (1.1) is generalized Ulam -Hyers stable if there exists $\theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\theta(0) = 0$ such that, for each solution $\mu \in PC^1(J, \mathbb{R})$ of inequality (4.1), there exists a solution $\vartheta \in PC^1(J, \mathbb{R})$ of (1.1) with

$$
|\mu(t) - \vartheta(t)| \le \theta(\varepsilon), \quad t \in J.
$$

Remark 4.3. A function $\mu \in PC^1(J, \mathbb{R})$ is a solution of inequality (4.1) if and only if there exist $\omega \in PC(J, \mathbb{R})$ and a sequence ω_k, $k = 1, \dots, m$ (which depend on μ) such that

1. $|\omega(t)| < \varepsilon$ and $|\omega_k(t)| < \varepsilon$ for $k = 1, \dots, m$;
2. $$
\begin{cases}
{}^{ABC}D_t^\alpha \mu(t) = \phi(t, \mu, {}^{ABC}D_t^\alpha \mu(t)) + \omega(t), & t \in J,\ 0 < \alpha < 1,\\
\Delta\mu(t_k) = I_k(\mu(t_k^-)) + \omega_k, & k = 1, \dots, m.
\end{cases}
\tag{4.2}
$$

Lemma 4.4. *If $\mu \in PC^1(J,\mathbb{R})$ is a solution of inequality (4.1), then μ satisfies the following integral inequality:*

$$\Bigg|\mu(t)-\mu_0-\frac{1-\alpha}{\beta(\alpha)}\tilde{\psi}(t)-\sum_{i=1}^{m}\frac{1-\alpha}{\beta(\alpha)}\tilde{\psi}(t_i)-\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\tilde{\psi}(s)\,ds$$
$$-\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}\tilde{\psi}(s)\,ds-\sum_{i=1}^{m}I_i(\mu(t_i^-))\Bigg|\le\delta\varepsilon,$$

where

$$\delta=\left(\frac{(1-\alpha)}{\beta(\alpha)}(m+1)+\frac{T^\alpha}{\beta(\alpha)\Gamma(\alpha)}(m+1)+m\right).$$

Proof. By Remark 4.3, μ satisfies (4.2). Using Lemma 3.2, its solution is

$$\begin{aligned}\mu(t)&=\mu_0+\frac{1-\alpha}{\beta(\alpha)}(\tilde{\psi}(t)+\omega(t))+\sum_{i=1}^{m}\frac{1-\alpha}{\beta(\alpha)}(\tilde{\psi}(t_i)+\omega(t_i))\\&+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(\tilde{\psi}(s)+\omega(s))\,ds\\&+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}(\tilde{\psi}(s)+\omega(s))\,ds+\sum_{i=1}^{m}I_i(\mu(t_i^-))+\omega_i,\end{aligned}$$

where $\tilde{\psi}(t)=\phi(t,\mu(t),\tilde{\psi}(t))$. Then the difference is bounded by

$$\frac{1-\alpha}{\beta(\alpha)}|\omega(t)|+\sum_{i=1}^{m}\frac{1-\alpha}{\beta(\alpha)}|\omega(t_i)|+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|\omega(s)|\,ds$$
$$+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}|\omega(s)|\,ds+\sum_{i=1}^{m}|\omega_i|\le\delta\varepsilon.$$

□

Theorem 4.5. *Assume that (H1) and (H2) hold and*

$$\Lambda^*=1-\frac{\lambda_1(1-\alpha)}{(1-\lambda_2)\beta(\alpha)}(m+1)>0.$$

Then the problem (1.1) is Ulam -Hyers stable.

Proof. Let $\mu \in PC^1(J,\mathbb{R})$ be a solution of inequality (4.1). Let ϑ denote the unique solution of the BVP

$$\begin{cases}{}^{ABC}D_t^\alpha\vartheta(t)=\phi(t,\vartheta,{}^{ABC}D_t^\alpha\vartheta(t)), & t\in J,\ 0<\alpha<1,\\ \Delta\vartheta(t_k)=I_k(\vartheta(t_k^-)), & k=1,\dots,m,\\ \vartheta(0)=\vartheta_0.\end{cases}$$

By Lemma 3.2, for $t\in(t_k,t_{k+1}]$,

$$\begin{aligned}\vartheta(t)&=\vartheta_0+\frac{1-\alpha}{\beta(\alpha)}\psi(t)+\sum_{i=1}^{m}\frac{1-\alpha}{\beta(\alpha)}\psi(t_i)+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\psi(s)\,ds\\&+\frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}\psi(s)\,ds+\sum_{i=1}^{m}I_i(\vartheta(t_i^-)),\end{aligned}$$

where $\psi(t) = \phi(t, \vartheta(t), \psi(t))$.

Now, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned}
|\mu(t) - \vartheta(t)| \leq & \Bigg| \mu(t) - \mu_0 - \frac{1-\alpha}{\beta(\alpha)}\tilde{\psi}(t) - \sum_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)}\tilde{\psi}(t_i) \\
& - \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\tilde{\psi}(s)\,ds - \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1}\tilde{\psi}(s)\,ds \\
& - \sum_{i=1}^{m} I_i(\mu(t_i^-)) \Bigg| \\
& + \frac{1-\alpha}{\beta(\alpha)}|\tilde{\psi}(t) - \psi(t)| + \sum_{i=1}^{m} \frac{1-\alpha}{\beta(\alpha)}|\tilde{\psi}(t_i) - \psi(t_i)| \\
& + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}|\tilde{\psi}(s) - \psi(s)|\,ds \\
& + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1}|\tilde{\psi}(s) - \psi(s)|\,ds \\
& + \sum_{i=1}^{m} |I_i(\mu(t_i^-)) - I_i(\vartheta(t_i^-))|.
\end{aligned}$$

Using the previous lemma, (H1) and (H2), we get

$$\begin{aligned}
|\mu(t) - \vartheta(t)| \leq \delta\varepsilon + \frac{\lambda_1(1-\alpha)}{(1-\lambda_2)\beta(\alpha)}(m+1)|\mu(t) - \vartheta(t)| + \frac{\lambda_1\alpha}{(1-\lambda_2)\beta(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|\mu(s) - \vartheta(s)|\,ds \\
+ \sum_{i=1}^{m} \kappa|\mu(t_i^-) - \vartheta(t_i^-)|.
\end{aligned}$$

Thus

$$|\mu(t) - \vartheta(t)| \leq \frac{\delta}{\Lambda^*}\varepsilon + \frac{\lambda_1\alpha}{\Lambda^*(1-\lambda_2)\beta(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|\mu(s) - \vartheta(s)|\,ds + \sum_{i=1}^{m} \frac{\kappa}{\Lambda^*}|\mu(t_i^-) - \vartheta(t_i^-)|.$$

Applying Lemma 2.8, we obtain

$$|\mu(t) - \vartheta(t)| \leq \frac{\delta}{\Lambda^*}\varepsilon \prod_{k=1}^{m} \left(1 + \frac{\kappa}{\Lambda^*}\right) \exp\left(\frac{\lambda_1 T^\alpha}{\Lambda^*(1-\lambda_2)\beta(\alpha)\Gamma(\alpha)}\right) = \rho\varepsilon,$$

where

$$\rho = \frac{\delta}{\Lambda^*}\left[\left(1 + \frac{\kappa}{\Lambda^*}\right) \exp\left(\frac{\lambda_1 T^\alpha}{\Lambda^*(1-\lambda_2)\beta(\alpha)\Gamma(\alpha)}\right)\right]^m.$$

Hence (1.1) is Ulam -Hyers stable. □

5 Examples

In this section, an example is provided to verify the validity of the investigated results.

Example 5.1. Consider the following fractional differential equation

$$\begin{cases} {}^{ABC}D^{1/2}\vartheta(t) = \frac{1}{5}\left(\sin(\vartheta(t)) + {}^{ABC}D^{1/2}\vartheta(t)\right), & t \in [0,2], \\ \Delta\vartheta(1) = \frac{1}{10}\vartheta(1^-), & t_1 = 1, \\ \vartheta(0) = \frac{1}{2}. \end{cases} \tag{5.1}$$

Here $\alpha = \frac{1}{2}$, $T = 2$, and

$$\phi(t, \vartheta, \psi) = \frac{1}{5}(\sin(\vartheta) + \psi).$$

Then for any $\vartheta, \bar{\vartheta}, \psi, \bar{\psi} \in \mathbb{R}$ and $t \in [0,2]$,

$$|\phi(t, \vartheta, \psi) - \phi(t, \bar{\vartheta}, \bar{\psi})| \leq \frac{1}{5}|\vartheta - \bar{\vartheta}| + \frac{1}{5}|\psi - \bar{\psi}|,$$

and

$$|I_1(\vartheta) - I_1(\psi)| \leq \frac{1}{10}|\vartheta - \psi|.$$

Thus $\lambda_1 = \lambda_2 = \frac{1}{5}$, $\kappa = \frac{1}{10}$.

We compute the contractivity condition:

$$\left(\frac{\lambda_1(1-\alpha)}{\beta(\alpha)(1-\lambda_2)} + \frac{\lambda_1 T^\alpha}{\beta(\alpha)\Gamma(\alpha)(1-\lambda_2)}\right)(m+1) + m\kappa$$

with $m = 1$, $\alpha = 1/2$, $T = 2$, $\beta(1/2) = 1 - \frac{1}{2} + \frac{1/2}{\Gamma(1/2)} = \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \approx 0.7821$. Numerically,

$$\frac{\frac{1}{5}\cdot\frac{1}{2}}{\beta(1/2)\cdot\frac{4}{5}} \approx 0.082, \quad \frac{\frac{1}{5}\cdot 2^{1/2}}{\beta(1/2)\cdot\Gamma(1/2)\cdot\frac{4}{5}} \approx 0.367,$$

so the sum is about 0.449, times $(m+1) = 2$ gives 0.898, plus $m\kappa = 0.1$, total about $0.998 < 1$. The condition is satisfied. Also

$$\Lambda^* = 1 - \frac{\lambda_1(1-\alpha)}{(1-\lambda_2)\beta(\alpha)}(m+1) = 1 - \frac{2\cdot 0.1}{0.8\cdot 0.7821} \approx 1 - 0.3196 = 0.6804 > 0.$$

Hence by Theorems 3.3 and 4.5, the problem has a unique solution and is Ulam -Hyers stable.

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Authors' contributions

Hadjer Belbali and Maamar Benbachir contributed equally to the conception, analysis, and writing of this manuscript. Both authors read and approved the final version.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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