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An optimal reinsurance problem in the Cramér-Lundberg model with investment and transaction costs

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Abstract. We study optimal proportional reinsurance strategies for minimizing the probability of ruin in an extended Cramér–Lundberg risk model with investment returns and fixed transaction costs. Under the assumption of cheap proportional reinsurance, stochastic control methods are used to derive the Hamilton–Jacobi–Bellman equation, reformulated as a Volterra integral equation of the second kind.

To obtain tractable solutions, a block-by-block numerical scheme based on Simpson's rule is developed to approximate survival probabilities under light-tailed and heavy-tailed claim distributions.

Ruin probabilities increase with retention, transaction costs, and volatility, and decrease with higher investment returns, reflecting strong sensitivity to financial and operational frictions, as confirmed numerically: increasing returns from 10% to 30% reduces the ruin probability at $u = 5$ from 0.2319 to 0.1626 under mixed exponential claims, while raising transaction costs from $k = 0$ to $k = 1$ increases it from approximately 0.09 to 0.15 under Pareto$(3, 2)$ claims.

The optimal retention level is $\alpha^* = 0$, although in practice a small positive retention may be more realistic.

Keywords: Probability of ruin, Optimal reinsurance, Stochastic control, Hamilton-Jacobi-Bellman equation, Volterra integral equation, Cramér-Lundberg model, Investments, Transaction costs.

2020 Mathematics Subject Classification: 45D05, 60G51, 91B05, 91G05, 93E20 MSC2020

1 Introduction

Insurance companies face significant uncertainty due to fluctuating claim sizes, which can adversely affect their financial position. To ensure long-term stability, insurers must maintain sufficient reserves to meet claims at any given time. A key metric in this context is the surplus process, which quantifies the excess of assets over liabilities. When the surplus falls below zero, the insurer is considered technically insolvent. The probability that the surplus becomes

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negative, known as the probability of ruin, has therefore become a critical risk indicator in actuarial research and insurance operations.

To mitigate the risk of insolvency, insurers employ various strategies, such as purchasing reinsurance and investing in financial assets. Reinsurance transfers a portion of the risk to a reinsurer, while investment activities can generate additional income. Reinsurance is assumed to reduce the overall risk exposure of the insurer, while investment has the potential to enhance wealth [18], thereby positively influencing the surplus process. However, both strategies involve costs and transaction costs in particular have a detrimental effect on the surplus process [5].

While most studies on optimal reinsurance and investment strategies in the Cramér-Lundberg model assume negligible transaction costs, this study addresses the more realistic case where such costs are present. By incorporating reinsurance premiums and fixed adjustment costs, the analysis captures the resulting complexities in decision-making. The model thus provides a more practical framework for determining optimal reinsurance under investment and transaction cost frictions.

Reinsurance has been extensively examined within the framework of the Cramér-Lundberg model, particularly regarding its effectiveness in reducing the probability of ruin and optimizing risk management strategies. Kasumo et al. [10] emphasized the role of reinsurance in enhancing insurer survival under both light- and heavy-tailed claim distributions. Cheng et al. [5] noted that while reinsurance contributes to financial stability, its associated costs must be carefully managed. Liang et al. [14] highlighted the importance of balancing cost and coverage, especially under large claims and fixed transaction costs. Zhang [19] further extended the literature by embedding reinsurance strategies in a multi-objective framework, aimed at minimizing ruin while maximizing utility.

Investment strategies also play a crucial role in ensuring financial stability within the Cramér-Lundberg framework. Liu and Yang [15] demonstrated that allocating surplus to financial markets can significantly reduce the probability of ruin. Duan and Liang [6] highlighted the benefits of integrated reinsurance and investment strategies. More complex settings involving stochastic interest rates and volatility were explored by Gao and Wang [7]. Liang et al. [14] also incorporated regime-switching effects, reflecting economic cycles.

Fixed transaction costs significantly influence decision-making in reinsurance and investment by introducing frictions that alter optimal control strategies. Brachetta and Ceci [3] discussed how such costs result in inaction, where insurers defer reinsurance or investment adjustments until surplus levels cross specific thresholds. Shen [17] and Zhang et al. [21] further demonstrated that both the magnitude and timing of financial interventions are critically shaped by transaction costs. These frictions induce non-continuous control policies, where strategic decisions occur in discrete bursts rather than gradual changes. Bei et al. [2] emphasized that fixed transaction costs, especially under stochastic volatility and interest rate models, make strategic solvency management increasingly complex. As a result, the insurer must balance the cost of acting against the risk of remaining passive, making fixed transaction costs a central consideration in practical solvency management.

Although substantial research has been conducted on optimal reinsurance and investment strategies within the classical Cramér–Lundberg framework, important limitations remain. Many existing studies examine reinsurance and investment decisions separately or neglect the impact of fixed transaction costs on insurer solvency. Furthermore, limited attention has been given to the combined effect of proportional reinsurance, investment returns, and transaction costs within a unified actuarial risk framework. Motivated by these limitations, this study investigates the joint effect of these components on ruin probabilities.

In this paper, we extend the Cramér-Lundberg model by incorporating proportional reinsurance, investment in both risky and risk-free assets, and fixed transaction costs. The objective is to identify the optimal reinsurance retention level $\alpha \in [0, 1]$ that minimizes the ultimate ruin probability. The problem is formulated as a stochastic control problem and analyzed using the Hamilton-Jacobi-Bellman (HJB) equation. To enable numerical computation, the resulting integro-differential equation is transformed into a Volterra integral equation of the second kind (VIE-2). To solve this integral equation, we apply a fourth-order block-by-block numerical method based on Simpson's rule. The method is tested under different claim size distributions, including exponential, mixed exponential and Pareto to reflect both light-tailed and heavy-tailed risks.

The remainder of this paper is organized as follows: Section 2 presents the mathematical formulation of the extended Cramér-Lundberg model, incorporating reinsurance, investment, and transaction costs. Section 3 describes the methodology used to transform the Volterra integro-differential equation (VIDE) and numerically solve the resulting integral equation. Section 4 presents and discusses the results for various claim size distributions and parameter values. Finally, Section 5 concludes the paper, offers practical recommendations based on the findings and points out future research directions.

2 Model formulation

The foundational assumption for the study is that all random variables and stochastic quantities are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}, \mathbb{P})$ satisfying the usual conditions, that is, the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ is right-continuous and $\mathbb{P}$-complete. This section presents the mathematical formulation of the extended Cramér-Lundberg model incorporating proportional reinsurance, investment and fixed transaction costs.

2.1 Model assumptions

The extended model developed in this study is based on the following assumptions:

1. The insurance company operates with a fixed premium rate p, determined using the expected claim size and a constant safety loading factor.

2. The premium charged by the insurer is calculated using the *Expected Value Premium Principle* given by
$$p = (1 + \theta)\,\mathbb{E}[X],$$
where $\mathbb{E}[X]$ is the expected claim amount and $\theta > 0$ is the loading factor which accounts for profit and risk margin.

3. The insurer purchases cheap proportional reinsurance, meaning that the insurer and reinsurer have identical safety loading $\eta = \theta$.

   This assumption serves as a benchmark case, leading to a tractable premium structure and allowing focus on the interaction between risk transfer, investment returns, and transaction costs. Relaxing to $\eta \neq \theta$ leads to a more general but analytically complex optimisation problem.

4. The reinsurance contract is of the quota-share type with a fixed retention level $\alpha \in (0,1]$. Excess-of-loss or other forms are not considered.

5. Ruin is defined as the first time the surplus process U_t becomes negative. No dividend payments are considered in this work.

6. Empirical parameter estimation is not performed. Instead, standard values from existing actuarial literature are adopted [10,12].

2.2 Cramér-Lundberg model

The Cramér-Lundberg surplus process is given by

$$U_t = u + pt - S_t, \quad t \geq 0. \tag{2.1}$$

The components of this model are: U_t is the surplus at time $t \geq 0$, $u > 0$ is the initial surplus (i.e., $U_0 = u$), $S_t = \sum_{i=1}^{N_t} X_i$ is a compound Poisson process representing the aggregate claim amount, where X_i are claim sizes with a general distribution $f(x)$ and mean $\mathbb{E}[X] = \mu$; N_t is a homogeneous Poisson process having constant intensity λ, and p is the premium income rate, defined as $p = (1+\eta)\lambda\mu$, where η is the safety loading.

2.3 Cramér-Lundberg model with proportional reinsurance

Under proportional reinsurance, the insurer retains a fixed fraction $\alpha \in (0,1]$ of each claim, while the reinsurer covers the remaining $1-\alpha$. The reinsurance premium p_r is calculated using a reinsurer's safety loading $\theta \geq 0$. The net premium retained by the insurer after reinsurance is

$$p^\alpha = \lambda\mu \left[(1+\eta) - (1+\theta)(1-\alpha)\right]. \tag{2.2}$$

Assuming the insurer purchases cheap reinsurance (i.e., $\eta = \theta$), the net premium simplifies to

$$p^\alpha = \lambda\mu(1+\eta)\left[1-(1-\alpha)\right] = \alpha\lambda\mu(1+\eta) = \alpha p.$$

Thus, the surplus process under proportional reinsurance is given by

$$U_t^\alpha = u + \alpha pt - \alpha S_t. \tag{2.3}$$

Without reinsurance, the insurer's surplus process can be expressed as

$$Y_t = pt - \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \quad Y_0 = 0, \tag{2.4}$$

where p is the constant premium rate and X_i represents the size of the i-th claim in the insurance portfolio.

If proportional reinsurance is introduced, the insurer retains only a fraction α of each claim and premium and the surplus process then becomes

$$Y_t^\alpha = \alpha pt - \sum_{i=1}^{N_t} \alpha X_i, \quad t \geq 0, \quad Y_0 = 0. \tag{2.5}$$

2.4 Investment process

The insurer invests in risky and risk-free assets and the risky asset Z_t is assumed to follow a geometric Brownian motion

$$dZ_t = Z_t(r\,dt + \sigma\,dW_t), \tag{2.6}$$

and the risk-free asset B_t grows at rate $r_0 \geq 0$ by the dynamics

$$dB_t = r_0 B_t\,dt. \tag{2.7}$$

Following Komunte et al [12], the total return from both risky and risk-free investments can be modelled as

$$R_t = rt + \sigma W_t + \sum_{i=1}^{N_{R,t}} Z_i, \quad R_0 = 0, \tag{2.8}$$

where the summation term represents jumps in return due to market shocks and W_t is a standard Brownian motion (Wiener process) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

In line with Paulsen et al. [16], we assume that no jump components occur (i.e., $\lambda_R = 0$). This simplification is adopted to retain a tractable continuous-time control formulation while capturing market fluctuations through the volatility parameter σ. Although jump effects are not explicitly modelled, scenarios with elevated volatility are considered in the numerical analysis to partially reflect periods of heightened market uncertainty, the return process simplifies to the Black-Scholes form

$$R_t = rt + \sigma W_t, \quad R_0 = 0, \tag{2.9}$$

which implies that one unit of capital invested at time zero grows to e^{rt} at time t under continuous compounding.

2.5 Cramér-Lundberg model with reinsurance and investment

When proportional reinsurance and investment are both active, the surplus process combines reinsured underwriting income and stochastic investment returns. Following Kasumo et al. [10], the surplus evolves according to the dynamics

$$dU_t^\alpha = dY_t^\alpha + U_{t^-}^\alpha\,dR_t, \tag{2.10}$$

where $U_{t^-}^\alpha$ is the surplus just before time t. Integrating the stochastic differential equation (2.10) yields

$$U_t^\alpha = u + Y_t^\alpha + \int_0^t U_{s^-}^\alpha\,dR_s. \tag{2.11}$$

2.6 Cramér-Lundberg model with fixed transaction costs

To reflect operational realism, a fixed transaction cost $k \geq 0$ is introduced, representing administrative and portfolio management expenses associated with maintaining the reinsurance and investment strategy. The parameter k is interpreted as an effective average cost rate, so that kt represents a continuous-time approximation of cumulative operational costs over the investment horizon, rather than discrete or per-adjustment expenditures. The linear cost structure is adopted as a reduced-form representation of operational frictions within the continuous-time framework. The controlled surplus process becomes

$$U_t^{\alpha,k} = u + Y_t^\alpha + \int_0^t U_{s^-}^{\alpha,k}\,dR_s - kt. \tag{2.12}$$

2.7 Hamilton-Jacobi-Bellman (HJB) framework

The primary objective of this study is to determine the optimal reinsurance strategy $\alpha \in [0,1]$ that minimizes the probability of ruin. The Hamilton-Jacobi-Bellman (HJB) equation offers a rigorous framework for addressing such stochastic control problems within the classical Cramér-Lundberg model. It has become a cornerstone in contemporary risk management literature and has been applied in several recent contributions, including Aljaberi et al. [1] and Zhang et al. [20].

This subsection focuses on the derivation of the HJB equation associated with the optimal control problem. By employing analytical methods, the value function is identified as a particular solution to the HJB equation, which in turn facilitates the development of an implementable numerical scheme for approximating both the value function and the corresponding optimal reinsurance strategy.

This study adopts an approach similar to that of Kasumo et al. [10], who derived the HJB equation under a risk model perturbed by diffusion. In that model, the surplus process was expressed as

$$P_t = pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_{P,t}} X_{P,i}, \ t \geq 0, \quad P_0 = 0, \tag{2.13}$$

where the Brownian term $\sigma_P W_{P,t}$ captures stochastic fluctuations in the insurance process, modelling the inherent randomness in surplus evolution. The model reduces to the classical Cramér-Lundberg framework when $\sigma_P = 0$, while $\sigma_P > 0$ reflects more realistic surplus behaviour under continuous random noise. In Kasumo et al. [10], the investment process is modelled as

$$R_t = rt + \sigma_R W_{R,t}, \quad R_0 = 0, \tag{2.14}$$

and the combined surplus solution is given by

$$y_t = \overline{R}_t \left(y + \int_0^t \overline{R}_s^{-1} \, dP_s \right), \tag{2.15}$$

where the stochastic exponential is

$$\overline{R}_t = \exp\left\{ \left(r - \frac{1}{2}\sigma_R^2 \right) t + \sigma_R W_{R,t} \right\}, \quad t \geq 0.$$

The infinitesimal generator of the process Y_t is:

$$\mathcal{L}g(y) = \frac{1}{2}(\sigma_R^2 y^2 + \sigma_p^2) g''(y) + (ry + p) g'(y) + \lambda \int_0^\infty [g(y-x) - g(y)] \, dF(x). \tag{2.16}$$

In this paper, we make the following simplifications and/or modifications to equation (2.16):

- Set $\sigma_R = \sigma$, the investment volatility;

- Set $\sigma_p = 0$, recovering the classical surplus model;

- Replace p with αp to reflect the effect of proportional reinsurance;

- Include a fixed transaction cost $k \geq 0$.

With these simplifications and/or modifications, the infinitesimal generator thus becomes

$$\mathcal{L}g(u) = \frac{1}{2}\sigma^2 u^2 g''(u) + (ru + \alpha p - k)g'(u) + \lambda \int_0^\infty [g(u - \alpha x) - g(u)]\, dF(x). \tag{2.17}$$

Applying the dynamic programming principle and Itô's Lemma, we derive the HJB equation for the survival probability $\phi(u)$ as

$$\sup_{\alpha \in [0,1]} \mathcal{L}^\alpha \phi(u) = 0. \tag{2.18}$$

Expanding this expression, we obtain

$$\sup_{\alpha \in [0,1]} \left\{ \frac{1}{2}\sigma^2 u^2 \phi''(u) + (ru + \alpha p - k)\phi'(u) + \lambda \int_0^\infty [\phi(u - \alpha x) - \phi(u)]\, dF(x) \right\} = 0. \tag{2.19}$$

If the optimal strategy $\alpha^* \in [0, 1]$ exists, then equation (2.19) reduces to a nonlinear Volterra integro-differential equation

$$\frac{1}{2}\sigma^2 u^2 \phi''(u) + (ru + \alpha p - k)\phi'(u) + \lambda \int_0^u [\phi(u - \alpha x) - \phi(u)]\, dF(x) = 0. \tag{2.20}$$

Solving the Volterra integro-differential equation (2.20) analytically is highly challenging due to the presence of both differential and integral terms. These features make closed-form solutions intractable for most claim size distributions, thereby necessitating the use of numerical approximation techniques.

3 Model solution and analysis

This section outlines the methodological framework adopted to solve the stochastic control problem described by the Hamilton-Jacobi-Bellman (HJB) equation. Due to the analytical complexity of the resulting Volterra integro-differential equation (VIDE), we begin by transforming it into a more tractable Volterra integral equation of the second kind (VIE-2) through successive integration by parts. This transformation facilitates the use of numerical methods, particularly the block-by-block Simpson's method, to approximate the survival probability function.

3.1 Transforming the VIDE into a VIE

The Volterra integro-differential equation (VIDE) presented in Equation (2.20) can be converted into a VIE-2 through successive integration by parts. This transformation is crucial for numerical approximation and serves as the foundation for the methods applied in this study.

Theorem 3.1. *The integro-differential equation (2.20) is equivalent to the Volterra integral equation of the second kind*

$$\phi(u) + \int_0^u K(u, x)\phi(x)\, dx = \gamma(u), \tag{3.1}$$

for $u \in [0, \infty)$, where $\phi : [0, \infty) \to \mathbb{R}$ is the unknown survival function, $K(u, x)$ is a known continuous kernel and $\gamma(u)$ is a continuous forcing function which is equally known.

Specifically, the kernel and forcing function are, respectively,

$$K(u,x) = 2 \cdot \frac{(2r-3\sigma^2+\lambda)x + (\alpha p - k) + \lambda G(u-\alpha x) - (r-\sigma^2+\lambda)u}{\sigma^2 u^2},$$

$$\gamma(u) = 2 \cdot \frac{\alpha p - k}{\sigma^2 u} \cdot \phi(0). \tag{3.2}$$

The kernel $K(u,x)$ represents the cumulative effect of claim arrivals, investment fluctuations, and transaction costs on the evolution of the surplus process and ruin probability

Proof. We begin by integrating both sides of Equation (2.20) over the interval $[0,z]$:

$$0 = \frac{1}{2}\int_0^z \sigma^2 u^2 \phi''(u)\,du + \int_0^z (ru+\alpha p - k)\phi'(u)\,du - \lambda \int_0^z \phi(u)\,du + \lambda \int_0^z \int_0^u \phi(u-\alpha x)\,dF(x)\,du.$$

Using the change of variables $v = u - \alpha x$, the double integral becomes a convolution term and the equation becomes

$$0 = \frac{1}{2}\sigma^2 z^2 \phi'(z) + \int_0^z \left[(r-\sigma^2)u + \alpha p - k\right]\phi'(u)\,du - \lambda \int_0^z \phi(u)\,du + \lambda \int_0^z \int_0^u \phi(v) f(u-v)\,dv\,du. \tag{3.3}$$

Integrating the third term by parts and simplifying yields

$$\begin{aligned} 0 = {} & \frac{1}{2}\sigma^2 z^2 \phi'(z) + \left[(r-\sigma^2)z + \alpha p - k\right]\phi(z) - (\alpha p - k)\phi(0) \\ & - (r-\sigma^2+\lambda)\int_0^z \phi(v)\,dv + \lambda \int_0^z \phi(v)F(z-v)\,dv. \end{aligned} \tag{3.4}$$

Now, integrating equation (3.4) again over $[0,u]$ with respect to z, we obtain

$$\begin{aligned} 0 = {} & \frac{1}{2}\int_0^u \sigma^2 z^2 \phi'(z)\,dz + \int_0^u \left[(r-\sigma^2)z + \alpha p - k\right]\phi(z)\,dz - (\alpha p - k)\int_0^u \phi(0)\,dz \\ & - (r-\sigma^2+\lambda)\int_0^u \int_0^z \phi(v)\,dv\,dz + \lambda \int_0^u \int_0^z \phi(v)F(z-v)\,dv\,dz. \end{aligned}$$

Applying Fubini's Theorem and simplifying, we arrive at

$$\begin{aligned} 0 = {} & \frac{1}{2}\sigma^2 u^2 \phi(u) - u(\alpha p - k)\phi(0) \\ & + \int_0^u \left[(2r-3\sigma^2+\lambda)x + (\alpha p - k) + \lambda G(u-\alpha x) - (r-\sigma^2+\lambda)u\right]\phi(x)\,dx, \end{aligned} \tag{3.5}$$

where $G(x) = \int_0^x xF(v)\,dv$ is the cumulative tail-weighted distribution function.

Dividing through by $\frac{1}{2}\sigma^2 u^2$ yields the VIE-2

$$\phi(u) + 2\int_0^u \frac{(2r-3\sigma^2+\lambda)x + (\alpha p - k) + \lambda G(u-\alpha x) - (r-\sigma^2+\lambda)u}{\sigma^2 u^2}\phi(x)\,dx = 2 \cdot \frac{\alpha p - k}{\sigma^2 u}\phi(0). \tag{3.6}$$

□

Theorem 3.2. *Let $\gamma \in C([c,d])$ and $K \in C([c,d] \times [c,d])$. Then the Volterra integral equation of the second kind*

$$\phi(x) = \gamma(x) + \int_c^x K(x,t)\phi(t)\,dt$$

has a unique solution $\phi \in C([c,d])$.

The existence and uniqueness of the solution follows from standard results for Volterra integral equations of the second kind under mild regularity conditions on $K(x,t)$ and $\gamma(x)$. For a proof of Theorem 3.2, see Brunner [4] and Kress [13].

3.2 Numerical approximation of the VIE

To approximate the survival probability $\phi(u)$, governed by the VIE-2

$$\phi(u) + \int_0^u K(u,s)\phi(s)\,ds = \gamma(u), \tag{3.7}$$

we apply the block-by-block (B-by-B) method in conjunction with Simpson's rule. This approach is valid under the assumption $r > \frac{1}{2}\sigma^2$, which ensures $\phi(u) > 0$ for all u. In contrast, if $r \leq \frac{1}{2}\sigma^2$, ruin is certain, i.e., $\phi(u) = 0$.
The integral equation is discretized as:

$$\phi_n + h\sum_{i=1}^{n} w_i K_{n,i}\phi_i = \gamma_n, \tag{3.8}$$

To approximate the survival probability $\phi(u)$, the interval $[0,U]$ is discretised as $u_n = nh$, where $h = U/N$ denotes the step size. The block-by-block Simpson method is then applied to the underlying Volterra integral equation, yielding a recursive computational scheme.
The integral term is approximated using Simpson's rule over pairs of subintervals, where h denotes the step size and $w_i \in \{1,4,2,\ldots,4,1\}$ are the corresponding Simpson quadrature weights. This leads to a local system of algebraic equations that can be expressed in matrix. Each resulting 2×2 linear system is solved directly using standard matrix techniques.
The solution is constructed in successive blocks $(\phi_{2m+1}, \phi_{2m+2})$, with midpoint values required in the quadrature approximated using quadratic interpolation based on previously computed nodal values.
The method is consistent with the classical block-by-block Simpson formulation (see Kasumo [9] and Paulsen et al. [16]). Numerical experiments indicate fourth-order convergence with respect to the step size h, which aligns with the theoretical accuracy properties of Simpson-based quadrature.

In the literature, the B-by-B method with Simpson's rule has been extended and applied to various problems. Examples include sixth-order methods for nonlinear Volterra systems [11] and stochastic surplus models with diffusion and risky investments [8]. The advantages of the B-by-B method include the following:

- Self-starting without need for special initialization;

- Compatibility with adaptive step-size; and

- High accuracy compared to step-by-step schemes.

This numerical scheme provides a robust framework for estimating ruin probabilities in insurance models with complex financial dynamics.

4 Results and discussion

This section presents numerical results obtained via the Simpson-based B-by-B method described in the previous section. Simulations were conducted under two contrasting claim-size models:

- A *Mixed Exponential distribution* to capture frequent and small to moderate claims; and

- A *Pareto distribution* to model rare, large claims.

The computed ruin probabilities are evaluated under multiple scenarios, including:

- The classical Cramér-Lundberg model; and

- Extended models incorporating proportional reinsurance, investment in both risk-free and risky assets, and fixed transaction costs incurred during capital adjustments.

Additionally, asymptotic ruin probabilities are examined to assess long-term solvency behaviour and to identify optimal reinsurance strategies under varying market conditions. The numerical scheme was implemented in `Python 3.13.3` (64-bit) using `NumPy`, `Pandas`, and `Matplotlib`, and simulations were run on an HP ProBook 440 G3 P247 PC with an Intel Core i5-6200U CPU at 2.30GHz with 8.0GB RAM and 64-bit operating system, x64-based processor (Windows 10).

4.1 Sensitivity analysis of ruin probabilities

This subsection explores the sensitivity of ruin probabilities to key financial controls. Results are presented for both light- and heavy-tailed claim distributions to assess how claim severity interacts with reinsurance, investment, and cost parameters.

Claim size models

Mixed Exponential Distribution. To reflect heterogeneous claim severity, the mixed exponential model combines two exponential components:

$$f(x) = q\mu_1 e^{-\mu_1 x} + (1-q)\mu_2 e^{-\mu_2 x}, \quad x \geq 0,$$

where $\mu_1 = 0.5$, $\mu_2 = 0.1$, and $q = 0.9$. This set-up favours small claims but retains a moderate likelihood of larger losses.
Pareto Distribution. To capture tail risk, the Pareto distribution with density

$$f(x) = \frac{\alpha\kappa^\alpha}{(\kappa + x)^{\alpha+1}}, \ x \geq 0$$

is used with parameter values $\alpha = 3$, $\kappa = 2$. This heavy-tailed distribution models catastrophic losses, where the mean and variance are both finite, but higher-order moments may not exist. To analyze the effects of proportional reinsurance, investment, and transaction costs, we implement a block-by-block numerical scheme to solve the associated Volterra integral equations. The analysis is structured into three cases: varying retention levels α, investment parameters

r, σ, and fixed transaction cost k. Ruin probabilities are computed using mixed exponential and Pareto claim distributions. Results are presented below for each case to demonstrate numerical outcomes.
Across all cases, the sensitivity of ruin probability to financial parameters is governed by the tail behaviour of the claim distribution. Mixed exponential claims produce a cumulative risk structure that is highly responsive to volatility and investment returns, whereas Pareto claims are dominated by rare but severe losses, resulting in lower sensitivity to financial conditions.

Case 1: Proportional reinsurance

To evaluate the effectiveness of proportional reinsurance on the insurer's risk exposure, we consider the classical Cramér-Lundberg surplus process under the assumption that both investment income and fixed transaction costs are absent. Specifically, the surplus process simplifies to:

$$U_t^{\alpha} = \alpha p t - \sum_{i=1}^{N_t} \alpha X_i, \quad U_0 = 0,$$

where $\alpha \in [0, 1]$ denotes the fraction of each claim retained by the insurer. The case $\alpha = 1$ represents full retention (no reinsurance), serving as a baseline for comparison.
The survival probability $\phi(u)$ satisfies a VIE-2 of the form:

$$\phi(u) + \int_0^u K(u, x)\phi(x)\, dx = \gamma(u),$$

where $K(u, x) = -\frac{\lambda \bar{F}(u-\alpha x)}{\alpha p}$ and $\gamma(u) = \phi(0)$. The kernel structure depends explicitly on α, allowing us to quantify how reinsurance retention influences the probability of ruin. Numerical results are presented. These illustrate the effect of varying the reinsurance retention level $\alpha \in [0, 1]$ on the ultimate ruin probability. Higher retention levels increase ruin probability, while greater risk transfer through reinsurance reduces solvency risk. This implies that insurers with low surplus positions should adopt lower retention levels to improve solvency stability and reduce default risk exposure.

Table 4.1: Ultimate ruin probability for Mixed Exponential distribution ($\mu_1 = 0.5$, $\mu_2 = 0.1$, $q = 0.9$) with varying retention levels α

u	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$
0	0.66670	0.66670	0.66670	0.66670	0.66670
4	0.15060	0.12590	0.10600	0.08990	0.07680
8	0.04640	0.03750	0.03160	0.02770	0.02520
12	0.02530	0.02290	0.02160	0.02090	0.02050
16	0.02120	0.02048	0.01996	0.01993	0.02020
20	0.02019	0.01992	0.02012	0.01984	0.02011

The results in Table 4.1 and Figure 4.1 demonstrate a clear monotonic relationship between the retention level α and the ultimate ruin probability $\psi(u)$. Specifically, higher retention levels (i.e., where the insurer bears a greater proportion of claim costs) lead to significantly higher ruin probabilities. Conversely, lower values of α representing greater risk transfer to the reinsurer substantially reduce the likelihood of ruin. This behaviour is consistent across all initial surplus levels u, and especially pronounced near $u = 0$, where reinsurance exerts the

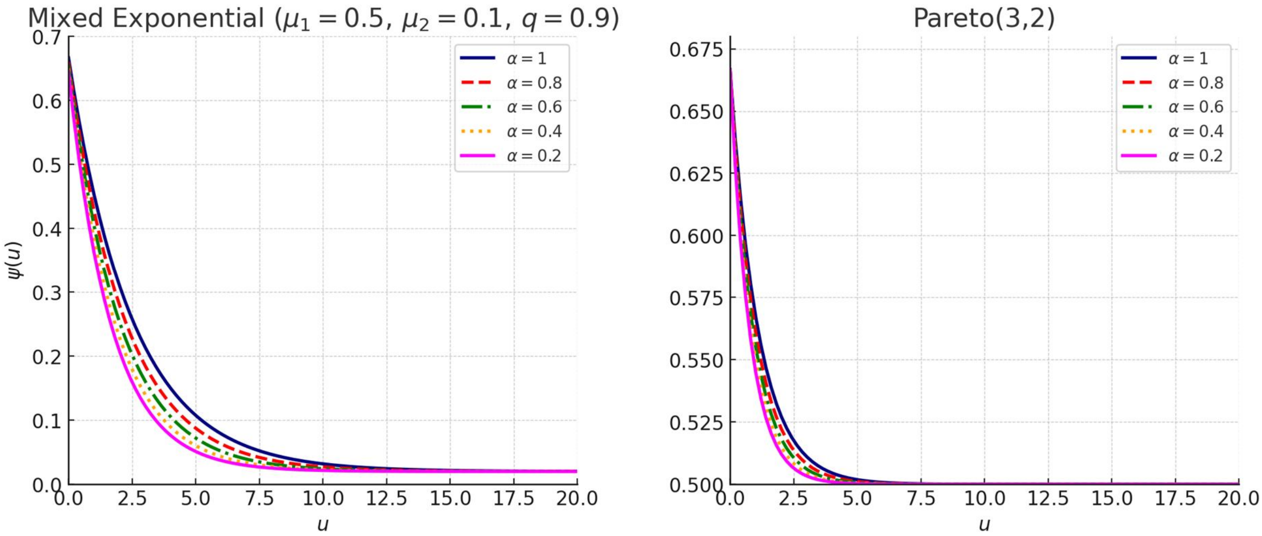

Figure 4.1: Impact of proportional reinsurance retention level $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ on ruin probability $\psi(u)$ for mixed exponential $(\mu_1 = 0.5, \mu_2 = 0.1, q = 0.9)$ and Pareto$(3, 2)$ claims, without investment and transaction costs. Parameters: $\lambda = 2$, $p = 6$, $\eta = 0.3$

greatest marginal benefit. These findings confirm the critical role of proportional reinsurance as an effective risk mitigation mechanism, applicable across both light- and heavy-tailed claim distributions.

Case 2: Effect of investment

Investment income significantly enhances an insurer's financial stability by offsetting claim payouts. This case examines how the *rate of return* and *investment volatility* impact the insurer's probability of ruin within the extended Cramér-Lundberg framework.

Impact of rate of return

Considering deterministic interest accumulation, the surplus process is

$$U_t = u + pt - \sum_{i=1}^{N_t} X_i + r \int_0^t U_s \, ds,$$

where r denotes the investment return rate. The survival probability $\phi(u)$ satisfies

$$(ru + p)\phi'(u) + \lambda \int_0^u [\phi(u - x) - \phi(u)] \, dF(x) = 0,$$

which transforms into the Volterra integral equation

$$\phi(u) = \frac{p}{ru + p}\phi(0) + \int_0^u \frac{r + \lambda \bar{F}(u - x)}{ru + p} \phi(x) \, dx,$$

with kernel and forcing function

$$K(u,x) = \frac{r + \lambda \bar{F}(u-x)}{ru+p}, \quad \gamma(u) = \frac{p}{ru+p}\phi(0).$$

Tables 4.2 and 4.3 display ruin probabilities $\psi(u) = 1 - \phi(u)$ for $r = 0.1, 0.2, 0.3$ under Mixed Exponential and Pareto(3,2) claims, respectively. Increasing r consistently reduces ruin risk.

Table 4.2: Ruin probabilities $\psi(u)$ at varying interest rates for Mixed Exponential

u	$r = 0.1$	$r = 0.2$	$r = 0.3$
0	0.6667	0.6667	0.6667
5	0.2319	0.1922	0.1626
10	0.0815	0.0509	0.0354
15	0.0256	0.0128	0.0077
20	0.0083	0.0031	0.0016

Table 4.3: Ruin probabilities $\psi(u)$ at varying interest rates for Pareto(3,2)

u	$r = 0.1$	$r = 0.2$	$r = 0.3$
0	0.6667	0.6667	0.6667
5	0.6015	0.5819	0.5623
10	0.5528	0.5234	0.5037
15	0.5112	0.4817	0.4621
20	0.4813	0.4518	0.4322

Impact of volatility

Table 4.4 below presents the effect of investment volatility on ruin probabilities.

Table 4.4: Ruin probabilities $\psi(u)$ at varying volatility σ for Mixed Exponential

u	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
0	0.6667	0.6667	0.6667
5	0.2519	0.2937	0.3391
10	0.1497	0.1946	0.2393
15	0.0983	0.1436	0.1899
20	0.0693	0.1064	0.1593

Tables 4.2, 4.3, and 4.4 show that higher investment returns reduce ruin probabilities for both claim types, while increased volatility, however, consistently raises ruin risk. Investment Returns and Volatility. Higher expected investment returns improve solvency by reducing ruin probability, whereas higher volatility has the opposite effect. This suggests that insurers should prioritise stable, diversified investment portfolios and avoid excessive exposure to high-volatility assets that may amplify ruin risk.

Case 3: Effect of transaction costs

We assess the impact of fixed transaction costs $k = 0, 1, 2$, where $k = 2$ approximates 30% of premium income ($p = 6$). Increasing fixed costs k raises ruin probabilities, as shown in Figure 4.2. Therefore fixed transaction costs require careful management in reinsurance and investment strategies. It should be noted that with $k = 0$ we have the model compounded only by reinsurance and investments. Increased transaction costs lead to higher ruin probabilities by eroding surplus over time. This indicates that actuarial valuation and solvency modelling should explicitly incorporate operational and administrative costs, as neglecting them may lead to overestimation of long-term financial stability.

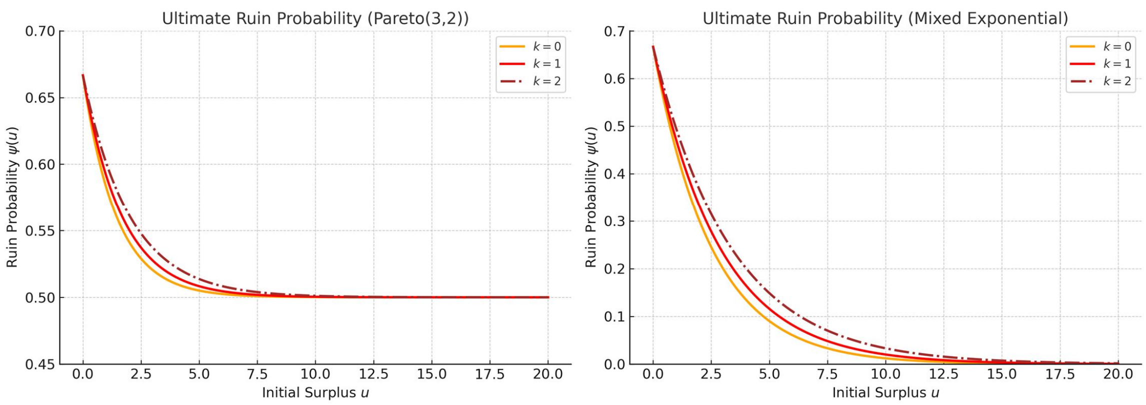

Figure 4.2: Ruin probabilities $\psi(u)$ under Pareto(3,2) and Mixed Exponential distributions for fixed transaction costs $k = 0, 1, 2$.

4.2 Asymptotic ruin probability and optimal retention level

The asymptotic approximation for the ruin probability with retention level α, initial surplus u, and net investment effect $\nu = r - k$ is given by:

$$\psi_\alpha(u) = \frac{\alpha}{\alpha + u + \nu}.$$

Minimizing $\psi_\alpha(u)$ with respect to α leads to the optimal retention:

$$\alpha^* = \begin{cases} 0, & \text{if } u + \nu > 0, \\ 1, & \text{if } u + \nu < 0. \end{cases}$$

Figure 4.3 illustrates these asymptotic ruin probabilities under Pareto claims with proportional reinsurance. The results confirm that full reinsurance ($\alpha^* = 0$) minimizes ruin probability when investment returns exceed transaction costs, supporting the strategy of maximizing risk transfer under heavy-tailed claims and moderate investment conditions.

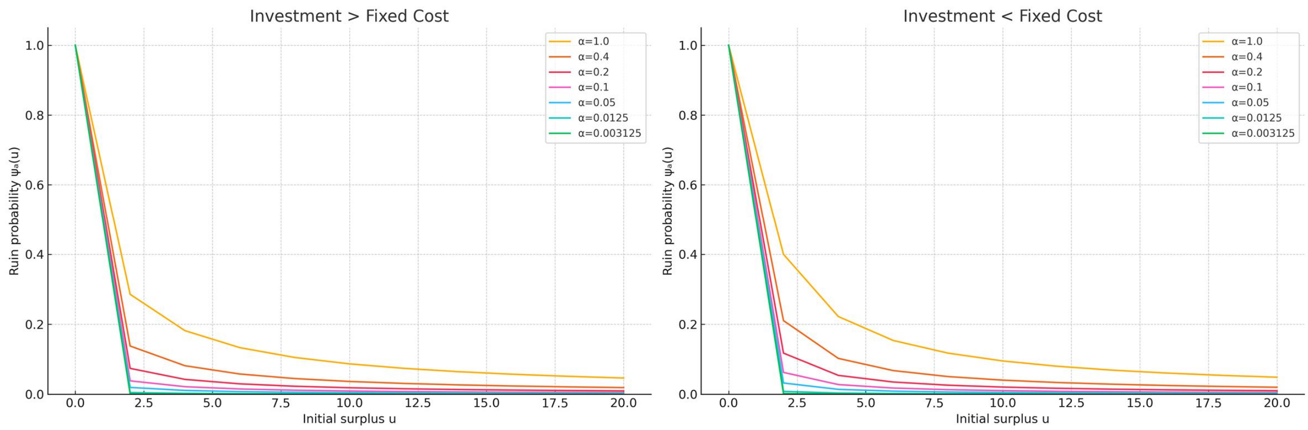

Figure 4.3: Asymptotic ruin probabilities for Pareto claims with proportional reinsurance with parameters $p = 6$, $\lambda = 2$, $r = 0.2$, $k_1 = 0.1$, $k_2 = 0.3$, and $\theta = \eta$.

4.3 Model validation and error analysis

To assess the accuracy of the block-by-block (B-by-B) numerical method, we validated the computed ruin probabilities against the exact solution for exponential claims. The results in Table 4.5 and Figure 4.4 show excellent agreement, with percentage errors generally below 2%. The method remains stable across all surplus levels and is especially accurate near $u = 0$, where the ruin probability is most sensitive.

These findings confirm the reliability of the B-by-B method for solving Volterra equations in risk modelling. Moreover, our results are consistent with benchmark studies by Kasumo [9] and Kasumo et al. [10], validating the method's accuracy against established Fortran-based solutions.

Table 4.5: Ultimate ruin probability in the classical model for $\text{Exp}(0.5)$

u	$\psi(u)$	$\psi_{0.01}(u)$	Error (%)
0	0.6666667	0.6666667	0.0000000
5	0.2897321	0.2857994	1.3574164
10	0.1259171	0.1244222	1.1871321
15	0.0547233	0.0537818	1.7205202
20	0.0237827	0.0230121	3.2397979
25	0.0103359	0.0099922	3.3250657
30	0.0044920	0.0044844	0.1677584
35	0.0019522	0.0019248	1.4009839
40	0.0008484	0.0008388	1.1308124
45	0.0003687	0.0003662	0.6833095
50	0.0001602	0.0001580	1.4290886

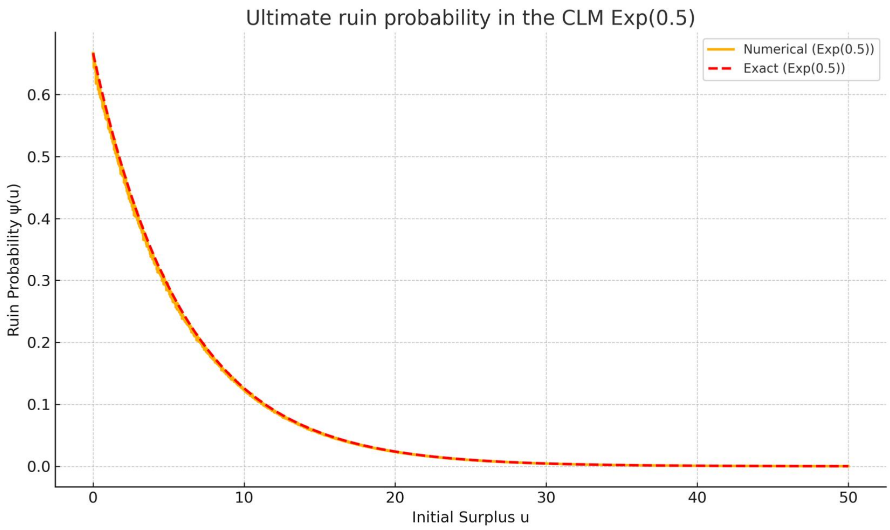

Figure 4.4: Ultimate ruin probability in the CLM for $\text{Exp}(0.5)$ claims.

5 Conclusion

This study developed a comprehensive model for ruin probabilities in an extended Cramér-Lundberg framework incorporating proportional reinsurance, investment, and fixed transaction costs. Both analytical and numerical analyses show that full reinsurance ($\alpha^* = 0$) minimizes the probability of ruin. Key financial factors influencing solvency include the investment return r, volatility σ, transaction costs k, and the retention level α. Numerical results align well with benchmark studies, validating the accuracy and robustness of the proposed approach. In particular, the findings demonstrate that higher investment returns reduce the probability of ruin, while fixed transaction costs necessitate more conservative reinsurance strategies for example reducing retention from $\alpha = 1.0$ to $\alpha = 0.2$ leads to approximately 40% reduction in ruin probability. Increasing volatility from $\sigma = 0.2$ to $\sigma = 0.4$ increases ruin probability from 0.15 to 0.24. Transaction costs consistently increase ruin probability.

These results highlight the significant impact of operational costs and investment dynamics on optimal reinsurance decisions.

While full reinsurance is theoretically optimal under the assumption of cheap reinsurance, practical considerations often require a low but non-zero retention level to balance cost and risk effectively. These insights are valuable for actuaries and risk managers in the design of sustainable insurance and reinsurance programs.

Based on the findings, insurers should prioritize high-return investment strategies and ensure diversification to mitigate volatility. Low retention levels are advisable when net investment gains are positive, and fixed transaction costs should be included in solvency assessments for realistic risk evaluation.

Future research may extend the present model in several directions. First, relaxing the assumption of cheap reinsurance by allowing different safety loadings ($\eta \neq \theta$) would lead to a more realistic premium structure and a higher-dimensional optimisation problem involving both retention and investment controls.

Second, the assumption of constant investment returns may be relaxed by introducing stochastic or regime-switching dynamics to better capture economic cycles and financial market uncertainty.

Third, the fixed transaction cost structure may be generalised using impulse control or singular stochastic control methods to model discrete adjustment decisions more realistically.

Fourth, alternative reinsurance arrangements such as excess-of-loss reinsurance can be incorporated to compare their efficiency with proportional reinsurance under different claim environments.

Finally, the model may be extended to incorporate regulatory frameworks such as IFRS 17 constraints to enhance practical actuarial applicability.

Declarations

Availability of data and materials

Data sharing is not applicable to this paper.

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Authors' contributions

PN wrote the manuscript, contributed to modelling and conducted data analyses. CK conceived the research and contributed to modelling and writing. Both authors read and approved the final manuscript.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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