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On some properties of Jacobsthal and Jacobsthal-Lucas numbers

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Abstract. In this study, we first develop some helpful generating functions for the Jacobsthal and Jacobsthal-Lucas numbers. We have discussed generating functions for $\mathbf{J}_{2n}$ and $\mathbf{j}_{2n}$ with just even and odd terms and generating functions for $\mathbf{J}_{2n+1}$, $\mathbf{j}_{2n+1}$, $\mathbf{J}_n\mathbf{J}_{n+1}$, $\mathbf{j}_n\mathbf{j}_{n+1}$, $\mathbf{J}_n^2$ and $\mathbf{j}_n^2$, where $\mathbf{J}_n$ and $\mathbf{j}_n$ are n^{th} Jacobsthal and Jacobsthal-Lucas numbers, respectively. Using these generating functions, we have found sum formulas for square Jacobsthal and Jacobsthal-Lucas numbers. Finally, we derive numerous useful convolution formulas including Jacobsthal numbers and similar expressions for Jacobsthal-Lucas numbers.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, Generating function, Convolution formulas.
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1 Introduction

The integer sequences such as Fibonacci, Pell, Jacobsthal, and others play a significant role in various branches of mathematics. The most famous integer sequences are the Fibonacci and Lucas sequences which are defined for $n \geq 2$, by the following second order recurrences

$$\mathbf{F}_n = \mathbf{F}_{n-1} + \mathbf{F}_{n-2}, \quad \mathbf{F}_0 = 0, \mathbf{F}_1 = 1, \tag{1.1}$$

and

$$\mathbf{L}_n = \mathbf{L}_{n-1} + \mathbf{L}_{n-2}, \quad \mathbf{L}_0 = 2, \mathbf{L}_1 = 1, \tag{1.2}$$

respectively. For more details and generalizations of these number sequences, the reader is referred to [10], [14], [15], [20].
In [9], Horadam discussed the generalized Horadam numbers as follows

$$w_{n+2} = pw_{n+1} - qw_n, \tag{1.3}$$

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with $w_0 = 2b$ and $w_1 = a + b$ for $n \geq 0$, where a, b, p and q are real numbers, usually integers. The explicit or Binet formula for these numbers are as follows

$$w_n = \frac{1}{\alpha - \beta}(A\alpha^n - B\beta^n), \tag{1.4}$$

where α and β are roots of characteristic equation $\lambda^2 - p\lambda + q = 0$, so

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2},$$
$$\beta = \frac{p - \sqrt{p^2 - 4q}}{2},$$

and

$$A = a + b - 2b\beta,$$
$$B = a + b - 2b\alpha.$$

For special cases, let $p = 1, q = -2, a = 1, b = 0$ and $p = 1, q = -2, a = 0, b = 1$, we get the following recurrences

$$\mathbf{J}_n = \mathbf{J}_{n-1} + 2\mathbf{J}_{n-2}, \quad \mathbf{J}_0 = 0, \mathbf{J}_1 = 1, n \geq 2, \tag{1.5}$$

and

$$\mathbf{j}_n = \mathbf{j}_{n-1} + 2\mathbf{j}_{n-2}, \quad \mathbf{j}_0 = 2, \mathbf{j}_1 = 1, n \geq 2, \tag{1.6}$$

these numbers are known as Jacobsthal and Jacobsthal-Lucas numbers, respectively, and have applications in number theory, combinatorics, coding theory and even electrical engineering. The characteristic equation of these sequences is $x^2 - x - 2 = 0$, along with the roots of $\alpha = 2$ and $\beta = -1$, thus by using these roots the Binet-like or explicit formulas are

$$\mathbf{J}_n = \frac{2^n - (-1)^n}{3}, \tag{1.7}$$

and

$$\mathbf{j}_n = 2^n + (-1)^n, \tag{1.8}$$

respectively. Table (1.1) includes the first few terms of $\mathbf{J}_n$ and $\mathbf{j}_n$.

n	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{J}_n$	0	1	1	3	5	11	21	43	85	171	341
$\mathbf{j}_n$	2	1	5	7	17	31	65	127	257	511	1027

Table 1.1: First few terms of $\mathbf{J}_n$ and $\mathbf{j}_n$.

For more details and relations about these sequences see [1], [3], [4], [7], [8], [9], [11], [16]. The Jacobsthal and Jacobsthal-Lucas sequences with negative indices are as

$$\mathbf{J}_{-n} = (-1)^{n+1}\frac{\mathbf{J}_n}{2^n} \tag{1.9}$$

and

$$\mathbf{j}_{-n} = (-1)^n\frac{\mathbf{j}_n}{2^n} \tag{1.10}$$

n	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{J}_{-n}$	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$-\frac{5}{16}$	$\frac{11}{32}$	$-\frac{21}{64}$	$\frac{43}{128}$	$-\frac{85}{256}$	$\frac{171}{512}$	$-\frac{341}{1024}$
$\mathbf{j}_{-n}$	2	$-\frac{1}{2}$	$\frac{5}{4}$	$-\frac{7}{8}$	$\frac{17}{16}$	$-\frac{31}{32}$	$\frac{65}{64}$	$-\frac{127}{128}$	$\frac{257}{256}$	$-\frac{511}{512}$	$\frac{1025}{1024}$

Table 1.2: First few terms of $\mathbf{J}_{-n}$ and $\mathbf{j}_{-n}$.

respectively, [18]. Table (1.2) includes some of these numbers with negative indices.

In reality, generating functions are the hooks for sequences, making them very useful tools for understanding sequence behavior. Therefore, it is simple to obtain the attributes of unknown sequences by using these functions. Therefore, finding many of those functions for a variety of sequences, including Jacobsthal and Jacobsthal-Lucas numbers, is our primary objective in this study. For more details about generating functions see [2], [6], [19].

The generating functions for Jacobsthal and Jacobsthal-Lucas numbers, $F(x)$ and $G(x)$, respectively, as follows

$$F(x) = \sum_{n=0}^{\infty} \mathbf{J}_n x^n = \frac{x}{1 - x - 2x^2}, \tag{1.11}$$

and

$$G(x) = \sum_{n=0}^{\infty} \mathbf{j}_n x^n = \frac{2 - x}{1 - x - 2x^2}. \tag{1.12}$$

The exponential generating functions are also important for sequences, for the Jacobsthal and Jacobsthal-Lucas numbers, these functions are as follows

$$f(x) = \sum_{n=0}^{\infty} \mathbf{J}_n \frac{t^n}{n!} = \frac{e^{2t} - e^{-t}}{3}, \tag{1.13}$$

and

$$g(x) = \sum_{n=0}^{\infty} \mathbf{j}_n \frac{t^n}{n!} = e^{2t} + e^{-t}, \tag{1.14}$$

respectively. In [12], Jhala et al. introduced and studied a new generalization of the Jacobsthal numbers and called these new numbers k-Jacobsthal numbers. For a positive integer k, the k-Jacobsthal numbers $\mathbf{J}_{k,n}$ are defined by the following recurrence relation

$$\mathbf{J}_{k,n+1} = k\mathbf{J}_{k,n} + 2\mathbf{J}_{k,n-1}, \ \mathbf{J}_{k,0} = 0, \mathbf{J}_{k,1} = 1, \tag{1.15}$$

where $n \geq 1$. In [5], Campos et al. defined the k-JacobsthalLucas numbers $j_{k,n}$ for $n \geq 1$ by the recurrence relation of

$$\mathbf{j}_{k,n+1} = k\mathbf{j}_{k,n} + \mathbf{j}_{k,n-1}, \ \mathbf{j}_{k,0} = 2, \mathbf{j}_{k,1} = k. \tag{1.16}$$

For $k = 1$, the usual Jacobsthal and Jacobsthal-Lucas numbers given by (1.5) and (1.6) can be obtained. In [13], the author find the generating function for $J_{k,an+r}$ as follows

$$\sum_{n=0}^{\infty} J_{k,an+r} x^n = \frac{J_{k,r} + (-2)^r J_{k,a-r} x}{1 - j_{k,a} x + (-2)^a x^2}, \tag{1.17}$$

where, $j_{k,n}$ is the n^{th} k-Jacobsthal-Lucas number. For the various values of a and r and assuming $k = 1$, the following generating functions for Jacobsthal numbers can be obtained

$$\sum_{n=0}^{\infty} J_{2n} x^n = \frac{x}{1 - 5x + 4x^2} \tag{1.18}$$

$$\sum_{n=0}^{\infty} J_{2n+1} x^n = \frac{1 - 2x}{1 - 5x + 4x^2}. \tag{1.19}$$

The generating functions given in (1.18) and (1.19) have all powers of x, but in the second section of this paper, by utilizing the generating functions given in (1.11) and (1.12), we will derive the generating functions just with even and odd powers of x for J_{2n}, j_{2n}, J_{2n+1} and j_{2n+1}, that is the generating functions of the form $\sum_{n=0}^{\infty} \mathbf{J}_{2n} x^{2n}$, $\sum_{n=0}^{\infty} \mathbf{j}_{2n} x^{2n}$, $\sum_{n=0}^{\infty} \mathbf{J}_{2n+1} x^{2n+1}$ and $\sum_{n=0}^{\infty} \mathbf{j}_{2n+1} x^{2n+1}$. In the rest of second section, we will obtain the generating functions for the squares and product of Jacobsthal numbers, that is the generating functions for J_n^2 and $J_n J_{n+1}$, as well as corresponding equations for the Jacobsthal-Lucas situation. The third section will look at summing formulae for the squares of the Jacobsthal and Jacobsthal-Lucas numbers using generating functions of sequences, which are cataloged in the Online Encyclopedia of Integer Sequences [17] as A024000 and A003683. In the final section, we will look at several important convolution formulas that use both Jacobsthal and Jacobsthal-Lucas numbers.

2 Generating functions with even and odd terms for $\mathbf{J}_{2n}$, $\mathbf{J}_{2n+1}$ and generating functions for $\mathbf{J}_n^2$, $\mathbf{J}_n\mathbf{J}_{n+1}$

In this section, we shall find generating functions involving even and odd terms for $\mathbf{J}_{2n}$ and $\mathbf{J}_{2n+1}$ and generating functions for $\mathbf{J}_n^2$ and $\mathbf{J}_n\mathbf{J}_{n+1}$ as well as related formulas for the Jacobsthal-Lucas situation.

Theorem 2.1. *The generating function of just even powers of x, for even terms of Jacobsthal sequence $\mathbf{J}_{2n}$ is given by*

$$F_e(x) = \sum_{n=0}^{\infty} \mathbf{J}_{2n} x^{2n} = \frac{x^2}{1 - 5x^2 + 4x^4}. \tag{2.1}$$

Proof. By using Equality (1.11) we have that

$$\frac{1}{2}(F(x) + F(-x)) = \sum_{n=0}^{\infty} \mathbf{J}_{2n} x^{2n},$$

on the other hand

$$F(x) + F(-x) = \frac{2x^2}{1 - 5x^2 + 4x^4},$$

therefore

$$\sum_{n=0}^{\infty} \mathbf{J}_{2n} x^{2n} = \frac{x^2}{1 - 5x^2 + 4x^4}.$$

□

Corollary 2.2. *The generating function of $\mathbf{j}_{2n}$ just with even powers of x is given as follows*

$$G_e(x) = \sum_{n=0}^{\infty} \mathbf{j}_{2n} x^{2n} = \frac{2 - 5x^2}{1 - 5x^2 + 4x^4}. \tag{2.2}$$

Proof. The proof can be done similarly to Theorem (2.1), by using (1.12). □

Theorem 2.3. *The generating function of odd powers of x, for odd terms of the Jacobsthal sequence $\mathbf{J}_{2n+1}$, is given by*

$$F_o(x) = \sum_{n=0}^{\infty} \mathbf{J}_{2n+1} x^{2n+1} = \frac{x - 2x^3}{1 - 5x^2 + 4x^4}. \tag{2.3}$$

Proof. By using Equality (1.11), we have that

$$\frac{1}{2}(F(x) - F(-x)) = \sum_{n=0}^{\infty} \mathbf{J}_{2n+1} x^{2n+1},$$

on the other hand

$$F(x) - F(-x) = \frac{2x - 4x^3}{1 - 5x^2 + 4x^4},$$

therefore

$$\sum_{n=0}^{\infty} \mathbf{J}_{2n+1} x^{2n+1} = \frac{x - 2x^3}{1 - 5x^2 + 4x^4}.$$

□

Corollary 2.4. *The generating function just with odd terms of Jacobsthal-Lucas numbers $\mathbf{j}_{2n+1}$ is given by*

$$G_o(x) = \sum_{n=0}^{\infty} \mathbf{j}_{2n+1} x^{2n+1} = \frac{x + 2x^3}{1 - 5x^2 + 4x^4}. \tag{2.4}$$

Proof. The proof can be done similar to Theorem (2.3), by using (1.12). □

Theorem 2.5. *The generating function for squares of Jacobsthal numbers $\mathbf{J}_n^2$, is as follows*

$$t(x) = \sum_{n=0}^{\infty} \mathbf{J}_n^2 x^n = \frac{x - 2x^2}{1 - 3x - 6x^2 + 8x^3}. \tag{2.5}$$

Proof. Let $t_n = \mathbf{J}_n^2$ for $n \geq 0$, then few terms of t_n are as follows

$$0, 1, 1, 9, 25, 121, 441, 1849, 7225, 29241, ...,$$

it can be easily verify that t_n follow the recurrence relation of

$$t_n = 3t_{n-1} + 6t_{n-2} - 8t_{n-3}\,; n \geq 3$$

with initial values of $t_0 = 0, t_1 = t_2 = 1$. Now, let $t(x)$ be generating function of t_n, then

$$\begin{aligned} t(x) &= 0 + x + x^2 + 9x^3 + 25x^4 + ... \\ -3xt(x) &= 0 - 3x^2 - 3x^3 - 27x^4 - ... \\ -6x^2t(x) &= 0 - 6x^3 - 6x^4 - ... \\ 8x^3t(x) &= 0 + 8x^4 + ... \end{aligned}$$

by adding these equations we obtain

$$t(x) = \frac{x - 2x^2}{1 - 3x - 6x^2 + 8x^3}.$$

□

Corollary 2.6. *The generating function of squares of Jacobsthal-Lucas numbers* $\mathbf{j}_n^2$ *is as follows*

$$s(x) = \sum_{n=0}^{\infty} \mathbf{j}_n^2 x^n = \frac{4 - 11x - 2x^2}{1 - 3x - 6x^2 + 8x^3}. \tag{2.6}$$

Proof. Let $s_n = \mathbf{j}_n^2$, then the sequence of s_n is as follows

$$4, 1, 25, 49, 289, 961, ...$$

This sequence follows the recurrence relation of

$$s_n = 3s_{n-1} + 6s_{n-2} - 8s_{n-3} \,; n \geq 3$$

with initial values of $s_0 = 4, s_1 = 1, s_2 = 25$. Let $s(x)$ be generating function of s_n, then

$$\begin{aligned} s(x) &= 4 + x + 25x^2 + 49x^3 + 289x^4 + ... \\ -3xs(x) &= -12x - 3x^2 - 75x^3 - 147x^4 - ... \\ -6x^2 s(x) &= -24x^2 - 6x^3 - 150x^4 - ... \\ 8x^3 s(x) &= 32x^3 + 8x^4 + ... \end{aligned}$$

by adding these equations we obtain

$$s(x) = \frac{4 - 11x - 2x^2}{1 - 3x - 6x^2 + 8x^3}.$$

□

Theorem 2.7. *The generating function for product of consecutive Jacobsthal numbers* $\mathbf{J}_n \mathbf{J}_{n+1}$, *is as follows*

$$p(x) = \sum_{n=0}^{\infty} \mathbf{J}_n \mathbf{J}_{n+1} x^n = \frac{x}{1 - 3x - 6x^2 + 8x^3}. \tag{2.7}$$

Proof. By using definition of Jacobsthal numbers we have

$$4\mathbf{J}_{n-1}^2 = \mathbf{J}_{n+1}^2 - 2\mathbf{J}_n \mathbf{J}_{n+1} + \mathbf{J}_n^2,$$

then

$$\begin{aligned} 2\sum_{n=0}^{\infty} \mathbf{J}_n \mathbf{J}_{n+1} x^n &= \sum_{n=0}^{\infty} \mathbf{J}_{n+1}^2 x^n + \sum_{n=0}^{\infty} \mathbf{J}_n^2 x^n - 4\sum_{n=0}^{\infty} \mathbf{J}_{n-1}^2 x^n \\ &= \frac{1}{x}\sum_{n=0}^{\infty} \mathbf{J}_n^2 x^n + \sum_{n=0}^{\infty} \mathbf{J}_n^2 x^n - 4\left(x\sum_{n=0}^{\infty} \mathbf{J}_n^2 x^n + \mathbf{J}_{-1}^2\right) \\ &= \left(\frac{1}{x} + 1 - 4x\right)\left(\frac{x - 2x^2}{1 - 3x - 6x^2 + 8x^3}\right) - 4\left(\frac{1}{4}\right) \\ &= \frac{2x}{1 - 3x - 6x^2 + 8x^3}. \end{aligned}$$

□

Corollary 2.8. *The generating function for the product of consecutive Jacobsthal-Lucas numbers* $\mathbf{j}_n \mathbf{j}_{n+1}$, *is as follows*

$$r(x) = \sum_{n=0}^{\infty} \mathbf{j}_n \mathbf{j}_{n+1} x^n = \frac{2 - x + 8x^2}{1 - 3x - 6x^2 + 8x^3}. \tag{2.8}$$

Proof. By using (1.6), we have that

$$4\mathbf{j}_{n-1}^2 = \mathbf{j}_{n+1}^2 - 2\mathbf{j}_n\mathbf{j}_{n+1} + \mathbf{j}_n^2,$$

then

$$\begin{aligned}
2\sum_{n=0}^{\infty} \mathbf{j}_n\mathbf{j}_{n+1}x^n &= \sum_{n=0}^{\infty} \mathbf{j}_{n+1}^2 x^n + \sum_{n=0}^{\infty} \mathbf{j}_n^2 x^n - 4\sum_{n=0}^{\infty} \mathbf{j}_{n-1}^2 x^n \\
&= \frac{1}{x}\left(\sum_{n=0}^{\infty} \mathbf{j}_n^2 x^n - \mathbf{j}_0^2\right) + \sum_{n=0}^{\infty} \mathbf{j}_n^2 x^n - 4\left(x\sum_{n=0}^{\infty} \mathbf{j}_n^2 x^n + \mathbf{j}_{-1}^2\right) \\
&= \left(\frac{1}{x} + 1 - 4x\right)\left(\frac{4 - 11x - 2x^2}{1 - 3x - 6x^2 + 8x^3}\right) - \frac{4}{x} - 4(\frac{1}{4}) \\
&= \frac{4 - 2x + 16x^2}{1 - 3x - 6x^2 + 8x^3}.
\end{aligned}$$

□

3 Summation formulas for the squares of Jacobsthal and Jacobsthal-Lucas numbers

In this section, by using generating functions, we will derive formulas for $\sum_{k=0}^{n} \mathbf{J}_n^2$ and $\sum_{k=0}^{n} \mathbf{j}_n^2$.

Theorem 3.1. *For the n^{th} Jacobsthal number $\mathbf{J}_n$, we have that*

$$\sum_{k=0}^{n} \mathbf{J}_k^2 = \frac{1}{9}(2^{n+2}\mathbf{J}_n + n). \tag{3.1}$$

Proof. Let a_n and b_n be sequences, entry A024000 and A003683 in [17], respectively, and defined as follows

$$a_n = 1 - n, \tag{3.2}$$

and

$$b_n = 2^{n-1}\mathbf{J}_n. \tag{3.3}$$

It can be easily verified that the generating functions of (3.2) and (3.3) are as follows

$$a(x) = \frac{1 - 2x}{(1 - x)^2},$$

and

$$b(x) = \frac{x}{(1 + 2x)(1 - 4x)},$$

respectively. Then

$$\begin{aligned}
a(x)b(x) &= \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} (1 - k)2^{n-k-1}\mathbf{J}_{n-k}\right) x^n.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
a(x)b(x) &= \frac{1-2x}{(1-x)^2}\frac{x}{(1+2x)(1-4x)} \\
&= \frac{\frac{1}{9}}{(1-x)^2} - \frac{\frac{1}{9}}{(1-x)} - \frac{\frac{4}{27}}{1+2x} + \frac{\frac{4}{27}}{1-4x} \\
&= \frac{1}{9}\sum_{n=0}^{\infty}(n+1)x^n - \frac{1}{9}\sum_{n=0}^{\infty}x^n - \frac{4}{27}\sum_{n=0}^{\infty}(-1)^n 2^n x^n + \frac{4}{27}\sum_{n\geq 0}4^n x^n \\
&= \frac{1}{9}\sum_{n=0}^{\infty}(2^{n+2}\mathbf{J}_n + n)x^n.
\end{aligned}
$$

Therefore

$$\sum_{k=0}^{n}(1-k)2^{n-k-1}\mathbf{J}_{n-k} = \frac{1}{9}(2^{n+2}\mathbf{J}_n + n),$$

few first terms of $\sum_{k=0}^{n}(1-k)2^{n-k-1}\mathbf{J}_{n-k}$ and $\sum_{k=0}^{n}\mathbf{J}_k^2$ are as follows

$$0, 1, 2, 11, 36, 157, 598, 2447, 9672, \ldots,$$

and

$$0, 1, 2, 11, 36, 157, 598, 2447, 9672, \ldots,$$

respectively. One can verify that both of the above sequences follow the recurrence relation of

$$x_n = 4x_{n-1} + 3x_{n-2} - 14x_{n-3} + 8x_{n-4}\ ; n \geq 4$$

with $x_0 = 0, x_1 = 1, x_2 = 2$ and $x_3 = 11$, then for every non-negative integer n,

$$\sum_{k=0}^{n}(1-k)2^{n-k-1}\mathbf{J}_{n-k} = \sum_{k=0}^{n}\mathbf{J}_k^2,$$

finally

$$\sum_{k=0}^{n}\mathbf{J}_k^2 = \frac{1}{9}(2^{n+2}\mathbf{J}_n + n).$$

□

Corollary 3.2. *Let $\mathbf{j}_n$ be the n^{th} Jacobsthal-Lucas number, then*

$$\sum_{k=0}^{n}\mathbf{j}_k^2 = n + 1 + \frac{1}{3}(1 + 2^{n+2}\mathbf{j}_n). \tag{3.4}$$

Proof. By using (1.8) we have that

$$
\begin{aligned}
\sum_{k=0}^{n}\mathbf{j}_k^2 &= \sum_{k=0}^{n}(2^k + (-1)^k)^2 \\
&= \sum_{k=0}^{n}4^k + 2\sum_{k=0}^{n}(-2)^k + \sum_{k=0}^{n}1,
\end{aligned}
$$

by using sums of geometric series and (1.8) we get the result. □

4 Some convolution formulas for Jacobsthal and Jacobsthal-Lucas numbers

This section will derive convolution formulae for Jacobsthal numbers using generating functions; the same methodology will also be applicable to Jacobsthal-Lucas numbers to get analogous formulas.

Theorem 4.1. *Let $\mathbf{J}_n$ be the n^{th} Jacobsthal number, then*

$$\sum_{k=0}^{n} \mathbf{J}_k \mathbf{J}_{n-k} = \frac{1}{9}\left(n\mathbf{j}_n - \mathbf{J}_n\right). \tag{4.1}$$

Proof. From Equality (1.11) we have that

$$F^2(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \mathbf{J}_k \mathbf{J}_{n-k}\right) x^n.$$

On the other hand,

$$\begin{aligned}
F^2(x) &= \left(\frac{-\frac{1}{3}}{x+1} + \frac{\frac{1}{3}}{-2x+1}\right)^2 \\
&= \frac{1}{9}\frac{1}{(x+1)^2} + \frac{1}{9}\frac{1}{(-2x+1)^2} - \frac{2}{27}\left(\frac{1}{x+1} + \frac{2}{-2x+1}\right) \\
&= \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n(n+1)x^n + \frac{1}{9}\sum_{n=0}^{\infty}(n+1)2^n x^n - \frac{2}{27}\left(\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} 2^{n+1}x^n\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{9}\left((n+1)((-1)^n + 2^n) - \frac{2}{3}\left((-1)^n + 2^{n+1}\right)\right)x^n,
\end{aligned}$$

therefore

$$\sum_{k=0}^{n} \mathbf{J}_k \mathbf{J}_{n-k} = \frac{1}{9}\left((n+\frac{1}{3})(-1)^n + (n-\frac{1}{3})2^n\right).$$

By using Equalities (1.7) and (1.8) we obtain

$$\sum_{k=0}^{n} \mathbf{J}_k \mathbf{J}_{n-k} = \frac{1}{9}\left(n\mathbf{j}_n - \mathbf{J}_n\right).$$

□

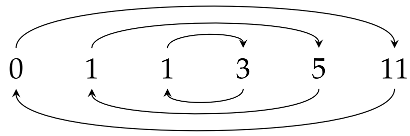
For example $\sum_{k=0}^{5} \mathbf{J}_k \mathbf{J}_{5-k} = 16$ as shown in Figure (4.1).

Corollary 4.2. *Let $\mathbf{j}_n$ be the n^{th} Jacobsthal-Lucas number, then*

$$\sum_{k=0}^{n} \mathbf{j}_k \mathbf{j}_{n-k} = (n+\frac{7}{3})\mathbf{j}_n - \frac{2}{3}(-1)^n. \tag{4.2}$$

Proof. From Equality (1.12) we have that

$$G^2(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \mathbf{j}_k \mathbf{j}_{n-k}\right) x^n.$$



$$0 \times 11 + 1 \times 5 + 1 \times 3 + 3 \times 1 + 5 \times 1 + 11 \times 0 = 16$$

Figure 4.1: Sum of $\sum_{k=0}^{5} \mathbf{J}_k \mathbf{J}_{5-k}$.

On the other hand,

$$\begin{aligned} G^2(x) &= \left(\frac{1}{x+1} - \frac{1}{2x-1}\right)^2 \\ &= \frac{1}{(x+1)^2} + \frac{1}{(2x-1)^2} + \frac{2}{3(x+1)} - \frac{4}{3(2x-1)} \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n + \sum_{n=0}^{\infty} (n+1) 2^n x^n + \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n x^n + \frac{4}{3} \sum_{n=0}^{\infty} 2^n x^n \end{aligned}$$

therefore

$$\sum_{k=0}^{n} \mathbf{j}_k \mathbf{j}_{n-k} = (n+1)(2^n + (-1)^n) + \frac{2}{3}(-1)^n + \frac{4}{3} 2^n.$$

By using (1.8) we get the result. □

Theorem 4.3. *The following convolution identity holds for Jacobsthal numbers*

$$\sum_{k=0}^{n} \mathbf{J}_{2k} \mathbf{J}_{2n-2k} = \frac{1}{9} \left((n+1)\mathbf{j}_{2n} - 2\mathbf{J}_{2n+2}\right). \tag{4.3}$$

Proof. By using the generating function given by Equality (2.1), we have

$$F_e(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathbf{J}_{2k} \mathbf{J}_{2n-2k} \right) x^{2n}.$$

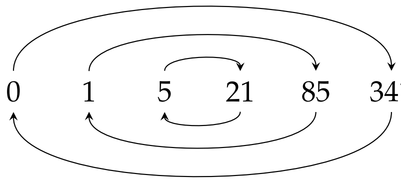
On the other hand

$$\begin{aligned} F_e(x)^2 &= \left(\frac{-\frac{1}{3}}{1-x^2} + \frac{\frac{1}{3}}{1-4x^2} \right)^2 \\ &= \frac{1}{9} \frac{1}{(1-x^2)^2} - \frac{2}{9} \frac{1}{(1-x^2)(1-4x^2)} + \frac{1}{9} \frac{1}{(1-4x^2)^2} \\ &= \frac{1}{9} \sum_{n=0}^{\infty} (n+1) x^{2n} - \frac{2}{9} \sum_{n=0}^{\infty} \mathbf{J}_{2n+2} x^{2n} + \frac{1}{9} \sum_{n=0}^{\infty} 4^n (n+1) x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{1}{9} \left((n+1)(4^n+1) - 2\mathbf{J}_{2n+2}\right) x^{2n}, \end{aligned}$$

therefore

$$\sum_{k=0}^{n} \mathbf{J}_{2k} \mathbf{J}_{2n-2k} = \frac{1}{9} \left((n+1)\mathbf{j}_{2n} - 2\mathbf{J}_{2n+2}\right).$$

□



$$0 \times 341 + 1 \times 85 + 5 \times 21 + 21 \times 5 + 85 \times 1 + 341 \times 0 = 380$$

Figure 4.2: Sum of $\sum_{k=0}^{5} \mathbf{J}_{2k}\mathbf{J}_{10-2k}$.

For example $\sum_{k=0}^{5} \mathbf{J}_{2k}\mathbf{J}_{10-2k} = 380$ as shown in Figure (4.2).

Corollary 4.4. *For Jacobsthal-Lucas numbers, the following identity holds*

$$\sum_{k=0}^{n} \mathbf{j}_{2k}\mathbf{j}_{2n-2k} = 2\mathbf{J}_{2n+2} + (n+1)\mathbf{j}_{2n}. \tag{4.4}$$

Proof. By using the generating function given by Equality (2.2), we have

$$G_e(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathbf{j}_{2k}\mathbf{j}_{2n-2k} \right) x^{2n}.$$

On the other hand

$$\begin{aligned} G_e(x)^2 &= \left(\frac{1}{1-x^2} + \frac{1}{1-4x^2} \right)^2 \\ &= \frac{1}{(1-x^2)^2} + \frac{2}{(1-x^2)(1-4x^2)} + \frac{1}{(1-4x^2)^2} \\ &= \sum_{n=0}^{\infty} (n+1)x^{2n} + 2\sum_{n=0}^{\infty} \mathbf{J}_{2n+2}x^{2n} + \sum_{n=0}^{\infty} 4^n(n+1)x^{2n} \\ &= \sum_{n=0}^{\infty} \left((n+1)(4^n+1) + 2\mathbf{J}_{2n+2} \right) x^{2n}, \end{aligned}$$

therefore

$$\sum_{k=0}^{n} \mathbf{j}_{2k}\mathbf{j}_{2n-2k} = (n+1)\mathbf{j}_{2n} + 2\mathbf{J}_{2n+2}.$$

□

Theorem 4.5. *For the n^{th} Jacobsthal number $\mathbf{J}_n$, we have*

$$\sum_{k=0}^{n} \mathbf{J}_{2k+1}\mathbf{J}_{2n-2k+1} = \frac{1}{9}\left((n+1)\mathbf{j}_{2n} + 4\mathbf{J}_{2n+2} \right). \tag{4.5}$$

Proof. The generating function for $\mathbf{J}_{2n+1}$ is the odd function of $F_o(x)$ which is given by (2.3) and

$$F_o(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathbf{J}_{2k+1}\mathbf{J}_{2n-2k+1} \right) x^{2n+2}.$$

And

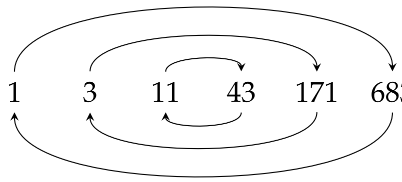
$$
\begin{aligned}
F_o(x)^2 &= \left(\frac{\frac{1}{3}x}{1-x^2} + \frac{\frac{2}{3}x}{1-4x^2} \right)^2 \\
&= \frac{1}{9}\frac{x^2}{(1-x^2)^2} + \frac{4}{9}\frac{x^2}{(1-x^2)(1-4x^2)} + \frac{4}{9}\frac{x^2}{(1-4x^2)^2} \\
&= \frac{1}{9}\sum_{n=0}^{\infty}(n+1)x^{2n+2} + \frac{4}{9}\sum_{n=0}^{\infty}\mathbf{J}_{2n+2}x^{2n+2} + \frac{4}{9}\sum_{n=0}^{\infty}4^n(n+1)x^{2n+2} \\
&= \sum_{n=0}^{\infty}\frac{1}{9}\left((n+1)(4^{n+1}+1) + 4\mathbf{J}_{2n+2}\right)x^{2n+2},
\end{aligned}
$$

therefore

$$
\sum_{k=0}^{n}\mathbf{J}_{2k+1}\mathbf{J}_{2n-2k+1} = \frac{1}{9}\left((n+1)\mathbf{j}_{2n+2} + 4\mathbf{J}_{2n+2}\right).
$$

□

For example $\sum_{k=0}^{5}\mathbf{J}_{2k+1}\mathbf{J}_{10-2k+1} = 3338$, as shown in Figure (4.3).



$$
1\cdot 683 + 3\times 171 + 11\times 43 + 43\times 11 + 171\times 3 + 683\times 1 = 3338
$$

Figure 4.3: Sum of $\sum_{k=0}^{5}\mathbf{J}_{2k+1}\mathbf{J}_{10-2k+1}$.

Corollary 4.6. *Let $\mathbf{j}_n$ be the n^{th} Jacobsthal-Lucas number, then*

$$
\sum_{k=0}^{n}\mathbf{j}_{2k+1}\mathbf{j}_{2n-2k+1} = (n+1)\mathbf{j}_{2n} - 4\mathbf{J}_{2n+2}. \tag{4.6}
$$

Proof. The generating function for $\mathbf{j}_{2n+1}$ is the odd function of $G_o(x)$ which is given by (2.4) and

$$
G_o(x)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathbf{j}_{2k+1}\mathbf{j}_{2n-2k+1}\right)x^{2n+2}.
$$

And

$$
\begin{aligned}
G_o(x)^2 &= \left(\frac{x}{1-x^2} + \frac{2x}{1-4x^2}\right)^2 \\
&= \frac{x^2}{(1-x^2)^2} + \frac{4x^2}{(1-x^2)(1-4x^2)} + \frac{4x^2}{(1-4x^2)^2} \\
&= \sum_{n=0}^{\infty}(n+1)x^{2n+2} + 4\sum_{n=0}^{\infty}\mathbf{J}_{2n+2}x^{2n+2} + \sum_{n=0}^{\infty}4^{n+1}(n+1)x^{2n+2} \\
&= \sum_{n=0}^{\infty}\left((n+1)(4^{n+1}+1) + 4\mathbf{J}_{2n+2}\right)x^{2n+2},
\end{aligned}
$$

therefore

$$\sum_{k=0}^{n} \mathbf{J}_{2k+1}\mathbf{J}_{2n-2k+1} = (n+1)\mathbf{j}_{2n+2} + 4\mathbf{J}_{2n+2}.$$

□

Theorem 4.7. *For the n^{th} Jacobsthal number $\mathbf{J}_n$, we have*

$$\sum_{k=0}^{n} \mathbf{J}_{2k}\mathbf{J}_{2n-2k+1} = \frac{1}{9}\left((n+1)\mathbf{j}_{2n+1} - \mathbf{J}_{2n+2}\right). \tag{4.7}$$

Proof. From (2.1) and (2.3), we have that

$$F_e(x)F_o(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \mathbf{J}_{2k}\mathbf{J}_{2n-2k+1}\right)x^{2n+1}.$$

On the other hand

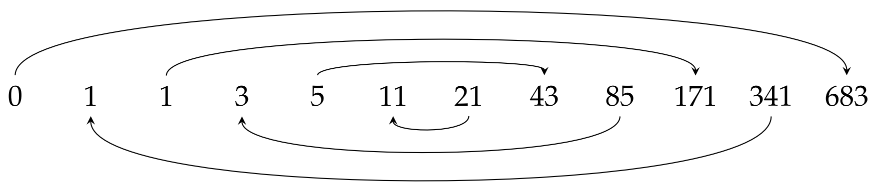
$$\begin{aligned}
F_e(x)F_o(x) &= -\frac{1}{9}\left(\frac{x}{(1-x^2)^2} - \frac{2x}{(1-4x^2)^2} + \frac{x}{(1-x^2)(1-4x^2)}\right)\\
&= -\frac{1}{9}\left(\sum_{n=0}^{\infty}(n+1)x^{2n+1} - 2\sum_{n=0}^{\infty}4^n(n+1)x^{2n+1} + \sum_{n=0}^{\infty}\mathbf{J}_{2n+2}x^{2n+1}\right)\\
&= -\frac{1}{9}\sum_{n=0}^{\infty}\left((n+1)(1-2\cdot 4^n) + \mathbf{J}_{2n+2}\right)x^{2n+1},
\end{aligned}$$

therefore

$$\sum_{k=0}^{n} \mathbf{J}_{2k}\mathbf{J}_{2n-2k+1} = \frac{1}{9}\left((n+1)\mathbf{j}_{2n+1} - \mathbf{J}_{2n+2}\right).$$

□

For example $\sum_{k=0}^{5} \mathbf{J}_{2k}\mathbf{J}_{10-2k+1} = 1213$, as shown in Figure (4.4).



$$0\cdot 683 + 341\cdot 1 + 1\cdot 171 + 85\cdot 3 + 5\cdot 43 + 21\cdot 11 = 1213$$

Figure 4.4: Sum of $\sum_{k=0}^{5} \mathbf{J}_{2k}\mathbf{J}_{10-2k+1}$.

Corollary 4.8. *For the n^{th} Jacobsthal-Lucas number $\mathbf{j}_n$, the following identity holds*

$$\sum_{k=0}^{n} \mathbf{j}_{2k}\mathbf{j}_{2n-2k+1} = (n+1)\mathbf{j}_{2n+1} + \mathbf{J}_{2n+2}. \tag{4.8}$$

Proof. From (2.2) and (2.4), we have that

$$G_e(x)G_o(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \mathbf{j}_{2k}\mathbf{j}_{2n-2k+1}\right)x^{2n+1}.$$

On the other hand

$$\begin{aligned} G_e(x)G_o(x) &= -\frac{x}{(1-x^2)^2} + \frac{2x}{(1-4x^2)^2} + \frac{x}{(1-x^2)(1-4x^2)} \\ &= -\sum_{n=0}^{\infty}(n+1)x^{2n+1} + 2\sum_{n=0}^{\infty}4^n(n+1)x^{2n+1} + \sum_{n=0}^{\infty}\mathbf{J}_{2n+2}x^{2n+1} \\ &= \sum_{n=0}^{\infty}\left((n+1)(2\cdot 4^n - 1) + \mathbf{J}_{2n+2}\right)x^{2n+1}, \end{aligned}$$

therefore

$$\sum_{k=0}^{n}\mathbf{j}_{2k}\mathbf{j}_{2n-2k+1} = (n+1)\mathbf{j}_{2n+1} + \mathbf{J}_{2n+2}.$$

□

Theorem 4.9. *Let* $\mathbf{J}_n$ *be the* n^{th} *Jacobsthal number, then*

$$\sum_{k=0}^{n}\mathbf{J}_k^2\mathbf{J}_{n-k}^2 = \frac{1}{81}\left(n2^{n+2}(-1)^n + (n+1)\mathbf{j}_{2n} - 2\right). \tag{4.9}$$

Proof. By using Equality (2.5) we have that

$$t(x)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathbf{J}_k^2\mathbf{J}_{n-k}^2\right)x^n.$$

And

$$\begin{aligned} t(x)^2 &= \frac{1}{81}\left(\frac{1}{(1-x)^2} - \frac{2}{1-x} + \frac{4}{(1+2x)^2} - \frac{4}{1+2x} + \frac{1}{(1-4x)^2}\right) \\ &= \frac{1}{81}\left(\sum_{n=0}^{\infty}(n+1)x^n - 2\sum_{n=0}^{\infty}x^n + 4\sum_{n=0}^{\infty}(n+1)(-1)^n2^nx^n - 4\sum_{n=0}^{\infty}(-1)^n2^nx^n + \sum_{n=0}^{\infty}(n+1)4^nx^n\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{81}\left(n2^{n+2}(-1)^n + (4^n+1)(n+1) - 2\right)x^n, \end{aligned}$$

therefore

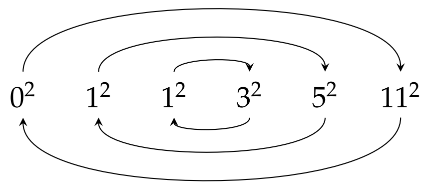
$$\sum_{k=0}^{n}\mathbf{J}_k^2\mathbf{J}_{n-k}^2 = \frac{1}{81}\left(n2^{n+2}(-1)^n + (n+1)\mathbf{j}_{2n} - 2\right).$$

□

For example $\sum_{k=0}^{5}\mathbf{J}_k^2\mathbf{J}_{5-k}^2 = 68$, as shown in Figure (4.5)

Corollary 4.10. *For the* n^{th} *Jacobsthal-Lucas number* $\mathbf{j}_n$*, the following identity holds*

$$\sum_{k=0}^{n}\mathbf{j}_k^2\mathbf{j}_{n-k}^2 = (n+1)\mathbf{j}_{2n} + 2^{n+2}\left((-1)^n(n+2) + \frac{1}{3}2^n\right) + \frac{2}{3}. \tag{4.10}$$



$$0^2 \cdot 11^2 + 1^2 \cdot 5^2 + 1^2 \cdot 3^2 + 3^2 \cdot 1^2 + 5^2 \cdot 1^2 + 11^2 \cdot 0^2 = 68$$

Figure 4.5: Sum of $\sum_{k=0}^{5} \mathbf{J}_k^2 \mathbf{J}_{5-k}^2$.

Proof. By using Equality (2.6) we have that

$$s(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathbf{j}_k^2 \mathbf{j}_{n-k}^2 \right) x^n.$$

And

$$\begin{aligned} s(x)^2 &= \left(\frac{1}{1-x} + \frac{2}{1+2x} + \frac{1}{1-4x} \right)^2 \\ &= \frac{1}{(1-x)^2} + \frac{\frac{2}{3}}{1-x} + \frac{4}{(1+2x)^2} + \frac{4}{1+2x} + \frac{1}{(1-4x)^2} + \frac{\frac{16}{3}}{1-4x} \\ &= \sum_{n=0}^{\infty} (n+1)x^n + \frac{2}{3} \sum_{n=0}^{\infty} x^n + 4 \sum_{n=0}^{\infty} (n+1)(-1)^n 2^n x^n + 4 \sum_{n=0}^{\infty} (-1)^n 2^n x^n + \sum_{n=0}^{\infty} (n+1) 4^n x^n \\ &+ \frac{16}{3} \sum_{n=0}^{\infty} 4^n x^n, \end{aligned}$$

therefore

$$\sum_{k=0}^{n} \mathbf{j}_k^2 \mathbf{j}_{n-k}^2 = (n+1)\mathbf{j}_{2n} + 2^{n+2} \left((-1)^n (n+2) + \frac{1}{3} 2^n \right) + \frac{2}{3}.$$

□

5 Conclusions

For comprehending sequences, generating functions are incredibly helpful tools. For other sequences, such as Jacobsthal and Jacobsthal-Lucas numbers, we investigated these functions. In addition to generating functions for squares and Jacobsthal and Jacobsthal-Lucas number products, a few generating functions that just include even and odd terms have been addressed. Additionally, we found a variety of useful summing formulas for squares of Jacobsthal and Jacobsthal-Lucas numbers utilizing generating functions. For these numbers, certain convolution formulas have been found using a very straightforward method, and we also offer schematic diagrams to help reader better understand these convolutions. Almost all of these result obtained by using generating functions quite easily.

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