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New framework for weighted Besov-Triebel-Lizorkin type spaces

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Abstract. In this paper, we introduce non-homogeneous weighted Besov-type spaces and non-homogeneous weighted Triebel-Lizorkin-type spaces. We then establish their characterizations in terms of local means and prove some remarkable embeddings. Moreover, we also construct Rychkov's linear, bounded universal extension operator for these spaces.

Keywords: Besov-type spaces, Triebel-Lizorkin-type spaces, extension operator, weighted function spaces, local means, Lipschitz domains.

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1 Introduction

The Besov and Triebel-Lizorkin spaces are classes of function spaces containing many well-known classical function spaces such as Lebesgue spaces L_p, Hardy spaces H_p [8], BMO spaces, Sobolev spaces, etc.

A comprehensive treatment of these function spaces and their history can be found in Triebel's monographs [29,30] and in the fundamental paper by M. Frazier and B. Jawerth [6].

Such spaces play an important role in various fields of analysis such as harmonic analysis and partial differential equations. For instance, Morrey [23] studied the local regularity of solutions of some partial differential equations in an appropriate space, called Morrey space. This local regularity of solutions is more precise than in the familiar Lebesgue spaces.

Over the past decades, classical operators in harmonic analysis, such as maximal, singular, and potential operators, have been extensively studied in both classical and generalized Morrey spaces. For more details, see [13–20,27] and the references therein.

In recent years, there has been increasing interest in a new family of function spaces introduced by Dachun Yang and Wen Yuan [36,37], called new classes of Besov and Triebel-Lizorkin spaces. These spaces unify and generalize many classical spaces such as Q spaces, Besov spaces, Triebel-Lizorkin spaces, Morrey spaces and Triebel-Lizorkin-Morrey spaces (see

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[32][Section 1.4] and references in [36]). From then on, these spaces received a lot of attention; various properties and characterizations of Besov-type and Triebel-Lizorkin-type spaces were studied in [21,22,33,34,38–40].

The present paper follows the spirit of Yasuo Komori and Satoru Shirai [18] and aims to further develop the ideas of Dachun Yang and Wen Yuan [36,37].

2 Some background tools

All notations used in this work are standard or will be defined when introduced.

A function w is termed a weight if it is almost everywhere positive and locally integrable in $\mathbb{R}^n$. Let w be a weight and $0 < p < \infty$, we say that $f \in L^p(w)$ if and only if

$$||f||_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

A non-negative locally integrable function w is said to be in the Muckenhoupt classes A_p, $1 < p < \infty$ if there exists a constant $C_p > 0$ such that for all cubes Q,

$$\frac{1}{|Q|} \int_Q w dy \left(\frac{1}{|Q|} \int_Q w^{1-p'} dy \right)^{p-1} \leq C_p \tag{2.2}$$

when $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and for $p = 1$,

$$\frac{1}{|Q|} \int_Q w dy \leq C_1 w(x), \tag{2.3}$$

for a.e. $x \in Q$, or equivalently $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator defined, for a locally integrable function f, by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{2.4}$$

where the supremum is taken over all cubes containing x. We set $A_\infty = \cup_{p \geq 1} A_p$.

We note that the classes A_p characterize the boundedness of the Hardy-Littlewood maximal operator M on the weighted Lebesgue spaces (see [9], Ch. IV). Namely,

$$M : L_p(w) \to L_p(w) \tag{2.5}$$

if and only if $w \in A_p$ when $1 < p < \infty$, and

$$M : L_1(w) \to L_{1,\infty}(w), \tag{2.6}$$

if and only if $w \in A_1$.
$L_{q,\infty}(w)$ denotes the space of all measurable functions f such that

$$\sup_{\lambda > 0} \left(w \left\{ x \in \mathbb{R}^n : f(x) > \lambda \right\} \right)^{\frac{1}{q}} < \infty.$$

We recall the following important properties of A_p classes, see for instance [1,7,9].

Lemma 2.1. *Let $w \in A_{r_0}$. Then there exist $0 < \delta < 1$ and $C > 0$ such that every time we have a measurable subset A of a cube Q, the following "δ-reverse doubling" inequality holds*

$$\frac{w(A)}{w(Q)} \leq C \left(\frac{|A|}{|Q|}\right)^{\delta} \tag{2.7}$$

and also the following "r_0-doubling" inequality holds

$$\frac{w(Q)}{w(A)} \leq C \left(\frac{|Q|}{|A|}\right)^{r_0}. \tag{2.8}$$

Moreover, if $1 < p < \infty$, $1 < q \leq \infty$ and $w \in A_p$, then there exists a positive constant C such that for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$, we have the well-known Fefferman-Stein vector-valued inequality, see [12],

$$\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} [Mf_k(x)]^q\right)^{p/q} w(x)dx \leq C \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} |f_k(x)|^q\right)^{p/q} w(x)dx. \tag{2.9}$$

Remark 2.2. If $w \in A_p$, then $w^{-1/(p-1)} \in A_{p/(p-1)}$ and satisfies the same "δ-reverse doubling" inequality. Throughout this work, w will be a fixed weight in A_∞ and we denote by r_0 the number $r_0 = \inf\{s > 0 : w \in A_s\}$. If we choose $0 < r < p/r_0,\ 0 < p < \infty$, then, in particular, $w \in A_{p/r}$ and $w^{-r/(p-r)} \in A_{p/(p-r)}$. The symbol δ represents the same reverse doubling constant for both w and $w^{-r/(p-r)}$.

The reverse condition is known as the A_∞-condition and the class of weights w satisfying the A_∞-condition is denoted by A_∞. It is well known that $A_\infty = \cup_{p\geq 1} A_p$, which motivates the notation A_∞, see for instance [9, Corollary 2.13, pp. 403-404].

We will use often the following well-known results.

Lemma 2.3. *Let f be any measurable function in $\mathbb{R}^n$ and $\lambda > n$. Then*

$$\int_{R^n} f(z) \left(1 + \frac{|x-z|}{t}\right)^{-\lambda} dz \leq Ct^n Mf(x)$$

for any $t > 0$.

Lemma 2.4. *Let $w \in A_p, 1 < p < \infty$. Then we have the following B_p-condition*

$$\int_{R^n} w(z) \left(1 + \frac{|x-z|}{t}\right)^{-np} dz \leq C \int_{|x-z|<t} w(z)dz$$

for any $t > 0$ and $x \in \mathbb{R}^n$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ with the classical topology generated by the family of semi-norms

$$||\nu||_{k,N} = \sup_{x\in\mathbb{R}^n} \sup_{|\beta|\leq N} (1+|x|)^k |\partial^\beta \nu(x)| \quad k, N \in \mathbb{N}_0, \quad \nu \in \mathcal{S}(\mathbb{R}^n).$$

The topological dual space, $\mathcal{S}'(\mathbb{R}^n)$, of $\mathcal{S}(\mathbb{R}^n)$ is the set of all continuous linear functionals $\mathcal{S}(R^n) \longrightarrow \mathbb{C}$ endowed with the weak $\star$-topology.

The Fourier transform, $\mathcal{F}\nu = \hat{\nu}$, of a Schwartz function ν is defined by

$$\hat{\nu}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \nu(x) dx.$$

The convolution of two functions $\nu, \mu \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\nu \star \mu(x) = \int_{\mathbb{R}^n} \nu(x-y)\mu(y) dy$$

and still belongs to $\mathcal{S}(\mathbb{R}^n)$.

The convolution operator can be extended to $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ via $\nu \star f(x) = \langle f, \mu(x - \cdot)\rangle$. It makes sense pointwise and is a C^∞ function in $\mathbb{R}^n$ of at most polynomial growth.

To simplify notation, we write often $\nu f = \nu \star f$. In some other situations, to avoid confusion, we keep the notation $\nu \star f$. We also set $\nu_t(x) = t^{-n}\nu(x/t)$ and $\nu_{2^{-j}}(x) = \nu_j(x)$. Recall that $\eta_{\nu,m} = 2^{n\nu}(1 + 2^{n\nu}|x|)^{-m}$, for any $x \in \mathbb{R}^n, \nu \in \mathbb{N}$ and $m > 0$. It is well known that $\eta_{\nu,m} \in L^1$ when $m > n$ and that $||\eta_{\nu,m}||_{L^1} = C_m$ is independent of ν.

Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line, while $C(a,b)$ denotes a positive constant depending on the parameters a, b. The symbols $A \preceq B$ means that $A \leq CB$ for some positive constant C. If $A \preceq B$ and $B \preceq A$, then we write $A \simeq B$. χ_E will denote the characteristic function of the set E. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we set Q_{jk} the dyadic cube $2^{-j}([0,1)^n + k)$. Let $\mathcal{Q} = \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ and $\mathcal{Q}_j := \{Q \in \mathcal{Q} : l(Q) = 2^{-j}\}$. For any $Q_j \in \mathcal{Q}$, we let j_Q be $-\log_2 l(Q)$, $l(Q)$ its side length, x_Q its lower left-corner $2^{-j}k$. We set also $j \vee 0 = \max(j, 0)$.

Definition 2.5. Let μ, ν be in the Schwartz space. We say a pair (μ, ν) of functions is admissible if $\mu, \nu \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp} \hat{\mu} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}, \quad \hat{\mu} \geq c > 0 \text{ when } |\xi| \leq \frac{5}{3}, \tag{2.10}$$

$$\operatorname{supp} \hat{\nu} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}, \quad \hat{\nu} \geq c > 0 \text{ when } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}. \tag{2.11}$$

Definition 2.6. Let $w \in A_\infty, 0 < p, q \leq \infty$, $0 \leq \tau < \infty$ and $\gamma \in \mathbb{R}$. The inhomogeneous Triebel-Lizorkin space $F^{\gamma,\tau}_{p,q,w}$ is the set of all distributions f (modulo polynomials) such that

$$||f||_{F^{\gamma,\tau}_{p,q,w}} \simeq \sup_{Q \in \mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q \vee 0}^{\infty} 2^{j\gamma q} |\nu_j f|^q \right)^{\frac{p}{q}} w(x) dx \right]^{\frac{1}{p}}. \tag{2.12}$$

The inhomogeneous Besov-Lipschitz space $B^{\gamma,\tau}_{p,q,w}$ is the set of all distributions f (modulo polynomials) such that

$$||f||_{B^{\gamma,\tau}_{p,q,w}} \simeq \sup_{Q \in \mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\sum_{j=j_Q \vee 0}^{\infty} \left(\int_Q |2^{j\gamma} \nu_j f|^p w(x) \right)^{\frac{q}{p}} dx \right]^{\frac{1}{q}}. \tag{2.13}$$

With standard modifications when $p = \infty$ or $q = \infty$. Here and in the rest of this paper ν_0 is understood as μ. For simplicity, in what follows, we use $A^{\gamma,\tau}_{p,q,w}$ to denote either $B^{\gamma,\tau}_{p,q,w}$ or $F^{\gamma,\tau}_{p,q,w}$. The most familiar case, the classical Besov spaces $B^{\gamma}_{p,q,w}$ and Triebel- Lizorkin spaces $F^{\gamma}_{p,q,w}$, are realized by letting $\tau = 0$. Moreover, it is well known that the spaces $A^{\gamma}_{p,q,w}$ are independent of the choices of ν, see, for example [2–4,6].

Proposition 2.1. Let B_J be any ball of $\mathbb{R}^n$ with radius 2^{-J}. In the definition of the spaces $A^{\gamma,\tau}_{p,q,w}$, if we replace the dyadic cubes Q by the balls B_J, then we obtain equivalent quasi-norms.

See Appendix for the proof.

3 Maximal functions and local means characterization

We define, for $\lambda > 0$, the Peetre maximal function $\nu^\star_{t,\lambda} f(x)$ by

$$\nu^\star_{t,\lambda} f(x) = \sup_{y\in\mathbb{R}^n} |\nu_t f(y)| \left(1 + \frac{|x-y|}{t}\right)^{-\lambda}.$$

We now present a fundamental characterization of the spaces under consideration.

Theorem 3.1. Let $\gamma \in \mathbb{R},\ \tau \geq 0,\ 0 < p,q \leq \infty$ and let (μ,ν) be a pair of admissible functions. If $\lambda > \frac{n}{\min(p,q)} + n\tau r_0$, then $f \in F^{\gamma,\tau}_{p,q,w}$ if and only if

$$||f||_{F^{\gamma,\tau}_{p,q,w}} \simeq \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q\vee 0}^{\infty} 2^{j\gamma q} |(\nu^\star_j f)_\lambda(x)|^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}},$$

and if $\lambda > \frac{n}{p} + n\tau r_0$, then $f \in B^{\gamma,\tau}_{p,q,w}$ if and only if

$$||f||_{B^{\gamma,\tau}_{p,q,w}} \simeq \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\left(\sum_{j=j_Q\vee 0}^{\infty} \int_Q |2^{j\gamma} (\nu^\star_j f)_\lambda(x)|^p\right)^{\frac{q}{p}} w(x)dx\right]^{\frac{1}{q}}.$$

To prove Theorem 3.1, we need the following well-known lemma, see for instance [31].

Lemma 3.1. *Let (μ,ν) be a pair of admissible functions, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $N \in \mathbb{N}$. Then for all $t \in [1,2]$, $N \geq \lambda$, $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$*

$$|(\nu^\star_{2^{-j}t} f)_\lambda(x)|^r \leq C \sum_{k=j}^{\infty} 2^{-(k-j)(Nr-n)} 2^{jn} \int_{\mathbb{R}^n} |\nu_{2^{-k}t} f(z)|^r (1 + |x-z|2^j)^{-\lambda r} dz$$

where r is an arbitrary fixed positive number and C is a positive constant independent of ν, f, j, x and t, but may depend on r.

Proof of Theorem 3.1. We only prove the first estimate of Theorem 3.1; the proof of the second estimate is similar.

Since $|\nu_j f(x)| \leq |\nu^\star_j f(x)|$, it is enough to prove

$$\sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q\vee 0}^{\infty} 2^{j\gamma q} |\nu^\star_{j,\lambda} f|^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}} \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}.$$

By Lemma 3.1 we see that

$$|(\nu_j^\star f)_\lambda(x)|^r \leq C\sum_{k=j}^{\infty}2^{-(k-j)(Nr-n)}2^{jn}\int_{\mathbb{R}^n}|\nu_k f(z)|^r(1+|x-z|2^j)^{-\lambda r}dz = I_{1j}(x)+I_{2j}(x),$$

with

$$I_{1j}(x)=\sum_{k=j}^{\infty}2^{-(k-j)(Nr-n)}2^{jn}\int_{|x-z|\leq 2^{-j_Q}}|\nu_k f(z)|^r(1+|x-z|2^j)^{-\lambda r}dz$$

and

$$I_{2j}(x)=\sum_{k=j}^{\infty}2^{-(k-j)(Nr-n)}2^{jn}\int_{|x-z|>2^{-j_Q}}|\nu_k f(z)|^r(1+|x-z|2^j)^{-\lambda r}dz.$$

Step 1. For $I_{1j}(x)$, by Lemma 2.3, we find

$$\begin{aligned} I_{1j}(x) \leq& C\sum_{k=j}^{\infty}2^{-(k-j)(Nr-n)}2^{jn}\int_{Q^\star}|\nu_k f(z)|^r(1+|x-z|2^j)^{-\lambda r}dz \\ \leq& C\sum_{k=j}^{\infty}2^{-(k-j)(Nr-n)}M\left(\chi_{Q^\star}|\nu_k f|^r\right)(x), \end{aligned}$$

where $Q^\star=c_nQ$ is chosen so that $\{|x-z|\leq 2^{-j_Q}\}\subset Q^\star$ for all $x\in Q$. Choose $0<r<\min(p,q)$, then by the vector Fefferman-Stein inequality and a discrete version of the Hardy inequality

$$\sum_j 2^{j\theta}\left(\sum_{k=j}^{\infty}|b_k|\right)^{\eta}\leq c\sum_j 2^{j\theta}|b_j|^{\eta}\ (\eta>0,\theta>0)$$

which we apply with $\theta=q(N+\gamma-n/r)$ and $\eta=q/r$, we have by taking $N>n/r-\gamma$

$$\begin{aligned} &\frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^{\infty}2^{j\gamma q}|I_{1j}(x)|^{q/r}\right)^{p/q}w(x)dx\right]^{1/p} \\ &\preceq \frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^{\infty}\left[\sum_{k=j}^{\infty}2^{-(k-j)(Nr-n+\gamma r)}2^{k\gamma r}M\left(\chi_{Q^\star}|\nu_k f|^r\right)(x)\right]^{q/r}\right)^{p/q}w(x)dx\right]^{1/p} \\ &\preceq \frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{k=j_Q\vee 0}^{\infty}\left[2^{k\gamma r}M\left(\chi_{Q^\star}|\nu_k f|^r\right)(x)\right]^{q/r}\right)^{p/q}w(x)dx\right]^{1/p} \\ &\preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}. \end{aligned}$$

Step 2. For $I_{2j}(x)$, in order to ensure the convergence of the series that will be considered below, we assume that $\lambda>n/\min(p,q)+n\tau r_0$. In this case there exist $0<r<\min(p,q)$ and $\epsilon>0$ such *that $n\tau r_0 < \epsilon < \lambda - n/r$. Then we have*

$$\begin{aligned}
I_{2j}(x) &= \sum_{k=j}^{\infty} 2^{-(k-j)(Nr-n)} 2^{jn} \int_{|x-z|>2^{-j_Q}} |\nu_k f(z)|^r (1+|x-z|2^j)^{-\lambda r} dz \\
&\preceq \sum_{k=j}^{\infty} 2^{-(k-j)(Nr-n)} \sum_{m=1}^{\infty} 2^{-m\epsilon r} 2^{jn} \int_{|x-z|<2^{-j_Q+m+1}} |\nu_k f(z)|^r (1+|x-z|2^j)^{-\lambda r + r\epsilon} dz \\
&\preceq \sum_{k=j}^{\infty} 2^{-(k-j)(Nr-n)} \sum_{m=1}^{\infty} 2^{-m\epsilon r} M(\chi_{2^{m+1}Q} |\nu_k f(x)|)^r.
\end{aligned}$$

Arguing as before we get

$$\begin{aligned}
&\frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q \vee 0}^{\infty} 2^{j\gamma q} |I_{2j}(x)|^{q/r} \right)^{p/q} w(x) dx \right]^{1/p} \\
&\preceq \left(\sum_{k=j}^{\infty} 2^{-(k-j)(Nr-n+r\gamma)} \sum_{m=1}^{\infty} 2^{-m\epsilon} \frac{[w(2^{m+1+n}Q)]^\tau}{[w(Q)]^\tau} \right)^{1/r} ||f||_{F^{\gamma,\tau}_{p,q,w}} \\
&\preceq \left(\sum_{m=1}^{\infty} 2^{-m\epsilon + mn\tau r_0} \right)^{1/r} ||f||_{F^{\gamma,\tau}_{p,q,w}} \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}},
\end{aligned}$$

since $\epsilon > n\tau r_0$. □

Definition 3.2. Let $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq \tau < \infty$ and $w \in A_\infty$. The space $L^\tau_{p,w}(l^q)$ is defined to be the set of all sequences $g = (g_j)_{j\in\mathbb{N}_0}$ of measurable functions on $\mathbb{R}^n$ such that

$$||(g_j)_{j\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q\vee 0}^{\infty} |g_j(x)|^q \right)^{p/q} w(x) dx \right]^{1/p} < \infty.$$

The space $l^q(L^\tau_{p,w})$ with $0 < p, q \leq \infty$ is defined to be the set of all sequences $g = (g_j)_{j\in\mathbb{N}_0}$ of measurable functions on $\mathbb{R}^n$ such that

$$||(g_j)_{j\in\mathbb{N}_0}||_{l^q(L^\tau_{p,w})} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\sum_{j=j_Q\vee 0}^{\infty} \left(\int_Q |g_j(x)|^p w(x) dx \right)^{q/p} \right]^{1/q} < \infty.$$

We need the following lemma which is a generalization of Lemma 2 in [24].

Lemma 3.3. *Let $r_0 \geq 1$, $w \in A_{r_0}$, $0 < q \leq \infty$, $0 \leq \tau < \infty$. Let $a > nr_0\tau$ and $b > 0$. For any sequence $g = \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$, consider*

$$G_j(x) = \sum_{\nu=0}^{j} 2^{-(j-\nu)a} g_\nu(x) + \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)b} g_\nu(x).$$

Then there exists a positive constant C, independent of g, such that for all $0 < p < \infty$,

$$||(G_j)_{j\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)} \leq C ||(g_j)_{j\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)} \tag{3.1}$$

and for all $0 < p \leq \infty$,

$$||(G_j)_{j\in\mathbb{N}_0}||_{l^q(L^\tau_{p,w})} \leq C ||(g_j)_{j\in\mathbb{N}_0}||_{l^q(L^\tau_{p,w})}. \tag{3.2}$$

Proof. Let us prove 3.1. The proof of 3.2 is similar and we omit the details. Let Q be a dyadic cube chosen arbitrarily and set

$$I_Q = \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q\vee 0} |G_j(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} = I_Q^a + I_Q^b$$

where

$$I_Q^a = \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q\vee 0}\left(\sum_{\nu=0}^{j} 2^{-(j-\nu)a}|g_\nu(x)|\right)^q\right)^{p/q} w(x)dx\right]^{1/p}$$

and

$$I_Q^b = \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q\vee 0}\left(\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)b}|g_\nu(x)|\right)^q\right)^{p/q} w(x)dx\right]^{1/p}.$$

Step1. We first assume that $0 < q \le 1$. Then by Young's inequality, for all $0 < \epsilon \le 1$ and $a_j \in C$,

$$\left(\sum_j |a_j|\right)^\epsilon \le \sum_j |a_j|^\epsilon \tag{3.3}$$

we have:

$$\begin{aligned} I_Q^b =& \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q\vee 0}\left(\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)b}|g_\nu(x)|\right)^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &\le \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q\vee 0}\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)bq}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &\le \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{\nu=j_Q\vee 0}\sum_{j=j_Q\vee 0}^{\nu} 2^{-(\nu-j)bq}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &\preceq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{\nu=j_Q\vee 0} |g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} = ||g||_{L^\tau_{p,w}(l^q)}. \end{aligned}$$

By using again Young's inequality, we find

$$\begin{aligned} I_Q^a &\le \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{\nu=0}^{\infty}\sum_{j=j_Q\vee\nu\vee 0}^{\infty} 2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &\le \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{\nu=0}^{j=j_Q\vee 0}\sum_{j=j_Q\vee 0}^{\infty} 2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &+ \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{\nu=j_Q\vee 0}^{\infty}\sum_{j=j_Q\vee\nu\vee 0}^{\infty} 2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &= B + A \end{aligned}$$

with

$$A = \frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{\nu=j_Q\vee 0}^{\infty}\sum_{j=j_Q\vee\nu\vee 0}^{\infty}2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{p/q}w(x)dx\right]^{1/p}$$

$$\leq \frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{\nu=j_Q\vee 0}^{\infty}\sum_{j=\nu}^{\infty}2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{p/q}w(x)dx\right]^{1/p}$$

$$\preceq ||g||_{L^\tau_{p,w}(l^q)}$$

and

$$B = \frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{\nu=0}^{j=j_Q\vee 0}\sum_{j=j_Q\vee 0}^{\infty}2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{p/q}w(x)dx\right]^{1/p}.$$

Using Hölder's inequality, the estimate 3.3 and choose $0<\epsilon<a-nr_0\tau$ *to get*

$$\left(\sum_{\nu=0}^{j_Q\vee 0}\sum_{j=j_Q\vee 0}^{\infty}2^{-(j-\nu)aq}|g_\nu(x)|^q\right)^{1/q} = \left(\sum_{j=j_Q\vee 0}^{\infty}2^{-jq}\sum_{\nu=0}^{j_Q\vee 0}2^{\nu q\epsilon}2^{(a-\epsilon)q}|g_\nu(x)|^q\right)^{1/q}$$

$$\preceq 2^{-(j_Q\vee 0)(a-\epsilon)}\sum_{\nu=0}^{j_Q\vee 0}2^{\nu(a-\epsilon)}|g_\nu(x)$$

$$\preceq 2^{-(j_Q\vee 0)(a-\epsilon)}\left(\sum_{\nu=0}^{j_Q\vee 0}2^{\nu(a-\epsilon)\theta}|g_\nu|(x)^\theta\right)^{1/\theta}, 0<\theta<1.$$

Thus

$$B \preceq \frac{2^{-(j_Q\vee 0)(a-\epsilon)}}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{\nu=0}^{j_Q\vee 0}2^{\nu(a-\epsilon)\theta}|g_\nu(x)|^\theta\right)^{p/\theta}w(x)dx\right]^{1/p}$$

$$\preceq \frac{2^{-(j_Q\vee 0)(a-\epsilon)}}{[w(Q)]^\tau}\left[\sum_{\nu=0}^{j_Q\vee 0}\left(\int_Q 2^{(a-\epsilon)\nu p}|g_\nu(x)|^p w(x)dx\right)^{\theta/p}\right]^{1/\theta}$$

$$\preceq 2^{-(j_Q\vee 0)(a-\epsilon)}\left[\sum_{\nu=0}^{j_Q\vee 0}\frac{2^{(a-\epsilon)\nu\theta}[w(2^{j_Q\vee 0-\nu}Q)]^{\tau\theta}}{[w(Q)]^{\tau\theta}}\frac{1}{[w(2^{j_Q\vee 0-\nu}Q)]^{\tau\theta}}\left(\int_{2^{j_Q\vee 0-\nu}Q}|g_\nu(x)|^p w(x)dx\right)^{\theta/p}\right]^{1/\theta}$$

$$\preceq 2^{-(j_Q\vee 0)(a-\epsilon)}\left[\sum_{\nu=0}^{j_Q\vee 0}\frac{2^{(a-\epsilon)\nu\theta}[w(2^{j_Q\vee 0-\nu}Q)]^{\tau\theta}}{[w(Q)]^{\tau\theta}}\right]^{1/\theta}||g||_{L^\tau_{p,w}(l^q)}$$

$$\preceq 2^{-(j_Q\vee 0)(a-\epsilon)}\left[\sum_{\nu=0}^{j_Q\vee 0}2^{(a-\epsilon)\nu\theta+(j_Q\vee 0-\nu)n\theta\tau r_0}\right]^{1/\theta}||g||_{L^\tau_{p,w}(l^q)}$$

$$\preceq ||g||_{L^\tau_{p,w}(l^q)}.$$

Step2. We assume now that $q>1$. *Then by the Jensen inequality, for all* $\epsilon>1, \delta>0$ *and* $a_j\in\mathbb{C}$,

$$\left(\sum_{j=0}^{\infty}2^{-\delta j}|a_j|\right)^\epsilon \preceq \sum_{j=0}^{\infty}2^{-\delta j}|a_j|^\epsilon,$$

we have

$$I_Q^b \leq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q\vee 0}\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)b}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p},$$

and

$$I_Q^a \leq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{\nu=0}^{\infty}\sum_{j=j_Q\vee\nu\vee 0}^{\infty} 2^{-(j-\nu)a}|g_\nu(x)|^q\right)^{p/q} w(x)dx\right]^{1/p}.$$

Then similar argument given in Step1 leads to,

$$I_Q^b \preceq ||g||_{L^\tau_{p,w}(l^q)}.$$

and

$$I_Q^a \preceq ||g||_{L^\tau_{p,w}(l^q)}.$$

□

Theorem 3.2. Let $R \in \mathbb{N}$ be such that $R > \gamma + nr_0\tau$. Assume that Ψ, ψ in the Schwartz space satisfies that, for all α with $|\alpha| \leq R$

$$\begin{aligned}
&\partial^\alpha \hat{\Psi}(0) = 0\\
&\hat{\psi}_0(0) > 0 \ \text{ if } |\xi| < K\epsilon\\
&\hat{\psi}(\xi) > 0 \ \text{ if } \frac{\epsilon}{2} < |\xi| < K\epsilon,
\end{aligned}$$

for some $\epsilon > 0$ and $1 < K \leq 2$. Then for $\lambda > \frac{n}{\min(p,q)} + n\tau r_0$

$$||(2^{j\gamma}(\Psi_j^\star f)_\lambda)_{j\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)} \preceq ||(2^{j\gamma}(\psi_j^\star f)_\lambda)_{j\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)}$$

and

$$||(2^{j\gamma}(\Psi_j^\star f)_\lambda)_{j\in\mathbb{N}_0}||_{l^q(L^\tau_{p,w})} \preceq ||(2^{j\gamma}(\psi_j^\star f)_\lambda)_{j\in\mathbb{N}_0}||_{l^q(L^\tau_{p,w})}.$$

Remark 3.4. This proves the independence of the spaces from a pair of admissible functions, since such a pair satisfies the conditions of Theorem 3.2 with $\epsilon = 6/5$ and $k = 25/18$.

For the proof we need the following lemma (see the proof of Theorem 3.4 in [11])

Lemma 3.5. *Let $R \in \mathbb{N}_0, \lambda > 0$ and $\gamma \in \mathbb{R}$. Assume that Ψ, ψ in the Schwartz space satisfies that, for all α with $|\alpha| \leq R$*

$$\begin{aligned}
&\partial^\alpha \hat{\Psi}(0) = 0\\
&\hat{\psi}_0(0) > 0 \ \text{ if } |\xi| < K\epsilon\\
&\hat{\psi}(\xi) > 0 \ \text{ if } \frac{\epsilon}{2} < |\xi| < K\epsilon,
\end{aligned}$$

for some $\epsilon > 0$ and $1 < K \leq 2$. Then for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$2^{j\gamma}(\Psi_j^\star f)_\lambda(x) \preceq \sum_{\nu=0}^{j} 2^{-(\nu-j)(R-\gamma)} 2^{\nu\gamma}(\psi_\nu^\star f)_\lambda(x) + \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)} 2^{\nu\gamma}(\psi_\nu^\star f)_\lambda(x)$$

Proof of Theorem 3.2. Since $R - \gamma > nr_0\tau$, then applying Lemma 3.3 and Lemma 3.5 to conclude.

4 Embeddings

Proposition 4.1. If $f \in F^{\gamma,\tau}_{p,q,w}$ and $h \in \mathcal{S}(\mathbb{R}^n)$. Then there exists $M > 0$ such that

$$| < f, h > | \preceq ||h||_{S_{M+1}} ||f||_{F^{\gamma,\tau}_{p,q,w}}.$$

If $f \in \mathcal{S}(\mathbb{R}^n)$ then there exists $M > 0$ such that

$$||f||_{F^{\gamma,\tau}_{p,q,w}} \preceq ||f||_{S_{M+1}}.$$

Hence

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{\gamma,\tau}_{p,q,w} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Similar results hold in the case of the Besov space.

We only prove Proposition 4.1 in the case of the Triebel space; for this we need the following results.

Lemma 4.1. *Let $\nu \in S(\mathbb{R}^n)$ with $\operatorname{supp} \hat{\nu} \subset \{\xi : |\xi| \leq 2\}$. Suppose $f \in F^{\gamma,\tau}_{p,q,w}$, then for any $0 < r < p/r_0$ and for any multi-indices β*

$$\sup_{z \in Q_{jk}} |\partial^\beta (\nu_j f(z))|^r \leq C 2^{jn} 2^{j|\beta|} 2^{-j\gamma r} ||f||^r_{F^{\gamma,\tau}_{p,q,w}}$$
$$\times \left(\int_{B(2^{-j}k, 2^{-j})} w dx \right)^{r\tau} \left(\int_{B(2^{-j}k, 2^{-j})} w^{-\frac{r}{p-r}} dx \right)^{1-r/p}.$$

Corollary 4.1.

$$||f||_{F^{\gamma,1/p-\tau}_{\infty,\infty,w}} = \sup_{Q \in \mathcal{Q}} \sup_{z \in Q} |Q|^{-\gamma/n} w(Q)^{1/p-\tau} |\nu_{\nu_{jQ}} f(z)| \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}.$$

In particular:

$$||f||_{F^{\gamma+n\tau-n/p}_{\infty,\infty}} \preceq ||f||_{F^{\gamma,\tau}_{p,q}}.$$

For comparison, see for instance [32, Proposition 2.6].

Proof. By Lemma 4.1 we have for all $z \in Q_{jk}$,

$$|\nu_{jQ} f(z)| \preceq |Q_{jk}|^{\gamma/n} \left(\int_{Q_{jk}} w dx \right)^{\tau - 1/p} ||f||_{F^{\gamma,\tau}_{p,q,w}}.$$

Hence

$$||f||_{F^{\gamma,1/p-\tau}_{\infty,\infty,w}} = \sup_{Q \in \mathcal{Q}} \sup_{z \in Q} |Q|^{-\gamma/n} w(Q)^{1/p-\tau} |\nu_{\nu_{jQ}} f(z)| \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}.$$

□

Proof of Lemma 4.1. Notice that $\operatorname{supp} \widehat{\nu_j f} \subset \{\xi : |\xi| \leq 2^{j+1}\}$. It follows from the proof of Lemma 2.4 in [6] that for any $r > 0$, for any multi-indices β and $N > n+1$, there exists $C = C_{r,M,\beta,N} > 0$ such that

$$\sup_{z \in Q_{jk}} |\partial^\beta (\nu_j f(z))|^r \leq C 2^{jn} 2^{j|\beta|} \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-N} \int_{Q_{j,k+l}} |\nu_j f(x)|^r dx. \tag{4.1}$$

Let $r > 0$ be such that $w \in A_{p/r}$. Hölder's inequality implies

$$\begin{aligned}\sup_{z\in Q_{jk}} |\partial^\beta(\nu_j f(z))|^r &\le C2^{jn}2^{j|\beta|}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\\ &\quad\times\Big(\int_{Q_{j,k+l}}|\nu_j f(x)|^p w dx\Big)^{r/p}\Big(\int_{Q_{j,k+l}} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}\\ &\le C2^{jn}2^{j|\beta|}2^{-j\gamma r}||f||^r_{F^{\gamma,\tau}_{p,q,w}}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\\ &\quad\times\Big(\int_{Q_{j,k+l}} w dx\Big)^{r\tau}\Big(\int_{Q_{j,k+l}} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}.\end{aligned}\tag{4.2}$$

Noting that $Q_{j,k+l} \subset B(2^{-j}k, \sqrt{n}2^{-j}(1+|l|))$. It follows that

$$\begin{aligned}&\Big(\int_{Q_{j,k+l}} w dx\Big)^{r\tau}\Big(\int_{Q_{j,k+l}} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}\\ &\le C(1+|l|)^{n\delta(r\tau+1-r/p)}\Big(\int_{B(2^{-j}k,\sqrt{n}2^{-j})} w dx\Big)^{r\tau}\Big(\int_{B(2^{-j}k,\sqrt{n}2^{-j})} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}.\end{aligned}$$

□

Corollary 4.2. Under the same assumptions in Lemma 4.1, there exists $N > 0$ such that for all j and k

$$\sup_{z\in Q_{jk}}|\nu_j f(z)|^r \le C2^{jn}2^{-j\gamma r}2^{-\min(j,0)N}||f||^r_{F^{\gamma,\tau}_{p,q,w}}\inf_{x\in Q_{jk}}(1+|x|)^N.\tag{4.3}$$

Consequently, if $j \ge 0$, then

$$\sup_{x\in\mathbb{R}^n}\frac{|\nu_j f(x)|}{(1+|x|)^N} \le 2^{j(n/r-\gamma)}||f||_{F^{\gamma,\tau}_{p,q,w}}\tag{4.4}$$

with $N = n(p\tau+1)/r$.

Proof. Fix $x \in Q_{jk}$. Then we have the following elementary inclusions

$$\begin{aligned}B(2^{-j}k,\sqrt{n}2^{-j}) \subset B(0,\sqrt{n}2^{-j}(1+|k|)) &\subset B(0,c_n2^{-j}(1+2^j|x|))\\ &\subset B(0,c_n2^{-\min(j,0)}(1+|x|)),\end{aligned}\tag{4.5}$$

for some constant $c_n > 1$. Applying Lemma 2.1 and 4.5, we obtain

$$\begin{aligned}\int_{B(2^{-j}k,\sqrt{n}2^{-j})} w dy &\preceq \int_{B(0,c_n2^{-\min(j,0)}(1+|x|))} w dy\\ &\preceq 2^{-\min(j,0)np/r}(1+|x|)^{np/r}\int_{B(0,1)} w dy\end{aligned}$$

and similarly

$$\int_{B(2^{-j}k,\sqrt{n}2^{-j})} w^{-\frac{r}{p-r}}dy \preceq 2^{-\min(j,0)np/(p-r)}(1+|x|)^{np/(p-r)}\int_{B(0,1)} w^{-\frac{r}{p-r}}dy.$$

Thus by Lemma 4.1

$$\sup_{z\in Q_{jk}}|\nu_j f(z)|^r \le C2^{jn}2^{-j\gamma r}2^{-\min(j,0)n(p\tau+1)}(1+|x|)^{n(p\tau+1)}||f||^r_{F^{\gamma,\tau}_{p,q,w}}.\tag{4.6}$$

□

Now we prove Proposition 4.1.

Proof. We first prove that $F^{\gamma,\tau}_{p,q,w} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. To this end, we need to prove that there exists an $M \in \mathbb{N}$ such that for all $f \in F^{\gamma,\tau}_{p,q,w}$ and $h \in \mathcal{S}(\mathbb{R}^n)$,

$$|<f,h>| \preceq ||h||_{S_{M+1}}||f||_{F^{\gamma,\tau}_{p,q,w}}.$$

Let (μ,ν) and (Ψ,ψ) be as in Theorem 3.1. The Calderon formula, see [32, Lemma 2.3]

$$f = \Psi \star \mu \star f + \sum_{j=1}^{\infty} \nu_j \star \psi_j \star f$$

together with [32, Lemma 2.4] and Corollary 4.2, lead to

$$\begin{aligned}
|<f,h>| &\leq \int_{\mathbb{R}^n} |\mu \star f(x)||\Psi \star h(x)|dx + \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\nu_j f(x)\psi_j h(x)|dx \\
&\preceq ||h||_{S_{M+1}} \sum_{j=0}^{\infty} 2^{-jM} \int_{\mathbb{R}^n} |\nu_j f(x)|(1+|x|^{-(n+M)})dx \\
&\preceq ||h||_{S_{M+1}}||f||_{F^{\gamma,\tau}_{p,q,w}} \sum_{j=0}^{\infty} 2^{-jM+jn(1/r-\gamma)} \int_{\mathbb{R}^n} (1+|x|)^{-M-n+n(p\tau+1)/r}dx.
\end{aligned}$$

Then choose $M > \max(n(1/r-\gamma), n(p\tau+1)/r)$ to conclude. To prove $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{\gamma,\tau}_{p,q,w}$, we fix $x \in Q = Q_{j_Q k_Q}$, then as seen before we have for some $c_n > 1$

$$\begin{aligned}
&Q \subset B(0, c_n 2^{-\min(j_Q,0)}(1+|x|)), \\
&\text{and } B(0,1) \subset B(0, c_n 2^{-\min(j_Q,0)}(1+|x|)).
\end{aligned}$$

Thus, if $w \in A_{r_0}$ then

$$\begin{aligned}
\frac{w(B(0,1))}{w(Q)} &\leq \frac{B(0, c_n 2^{-\min(j_Q,0)}(1+|x|)))}{w(Q)} \\
&\preceq 2^{-\min(j_Q,0)nr_0}(1+|x|)^{nr_0} 2^{nj_Q r_0}.
\end{aligned}$$

Hence

$$2^{\min(j_Q,0)nr_0} 2^{-nj_Q r_0}(1+|x|)^{-nr_0} \preceq w(Q). \tag{4.7}$$

Assume first that $j_Q > 0$ and choose $r \in (0, \min(1,p,q))$. Then using [32, Lemma 2.4], to obtain

$$\begin{aligned}
&\frac{1}{[w(Q)]^{\tau}} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{j\gamma q} |\nu_j \star f(x)|^q \right)^{p/q} w(x)dx \right]^{1/p} \\
&\preceq ||f||_{S^{M+1}} \frac{1}{[w(Q)]^{\tau}} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{j\gamma q} 2^{-jMq}(1+|x|)^{-(M+n)q} \right)^{p/q} w(x)dx \right]^{1/p} \\
&\preceq ||f||_{S^{M+1}} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{-j(M-\gamma)r} \right)^{p/r} 2^{nj_Q r_0 p\tau}(1+|x|)^{-(M+n)p+nr_0 p\tau} w(x)dx \right]^{1/p} \\
&\preceq ||f||_{S^{M+1}} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{-j(M-\gamma)r} \right)^{p/r} 2^{nj_Q r_0 p\tau}(1+|x|)^{-(M+n)p+nr_0 p\tau} w(x)dx \right]^{1/p}.
\end{aligned}$$

Choose now $M > \gamma + n\tau r_0$, to get

$$\frac{1}{[w(Q)]^{\tau}} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{j\gamma q} |\nu_j \star f(x)|^q \right)^{p/q} w(x)dx \right]^{1/p}$$
$$\preceq ||f||_{S^{M+1}} 2^{-j_Q(M-\gamma-nr_0\tau)} \left[\int_Q (1+|x|)^{-(M+n)p+nr_0p\tau} w(x)dx \right]^{1/p}.$$

If $(M+n)p - nr_0p\tau > nr_0$ then since $w \in A_{r_0}$ we have

$$\int_Q (1+|x|)^{-(M+n)p+nr_0p\tau} w(x)dx$$
$$\leq \int_{\mathbb{R}^n} (1+|x|)^{-(M+n)p+nr_0p\tau} w(x)dx \preceq \int_{B(0,1)} w(x)dx.$$

We conclude, by choosing $M > \max(0, n - nr_0/p + n\tau r_0, \gamma + n\tau r_0)$, that

$$\frac{1}{[w(Q)]^{\tau}} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{j\gamma q} |\nu_j \star f(x)|^q \right)^{p/q} w(x)dx \right]^{1/p} \preceq ||f||_{S^{M+1}}.$$

On the other hand if $j_Q \leq 0$, then we have by 4.7

$$(1+|x|)^{-nr_0} \preceq w(Q).$$

Arguing as before to get

$$\frac{1}{[w(Q)]^{\tau}} \left[\int_Q \left(\sum_{j=0}^{\infty} 2^{j\gamma q} |\nu_j \star f(x)|^q \right)^{p/q} w(x)dx \right]^{1/p}$$
$$\preceq ||f||_{S^{M+1}} \left[\int_Q \left(\sum_{j=0}^{\infty} 2^{-j(M-\gamma)r} \right)^{p/r} (1+|x|)^{-(M+n)p+nr_0p\tau} w(x)dx \right]^{1/p} \preceq ||f||_{S^{M+1}}.$$

□

The following proposition can be obtained by arguing as in the proof of Proposition 4.1 given in [35].

Proposition 4.2. Let $\tau, \gamma \in \mathbb{R}, p, q \in (0, \infty]$ and $\epsilon \in (0,1)$.

1. The space $A^{\gamma,\tau}_{p,q_1,w}$ is monotone with respect to q, namely, if $q_1 \leq q_2$, then $A^{\gamma,\tau}_{p,q_1,w} \hookrightarrow A^{\gamma,\tau}_{p,q_2,w}$.
2. The space $A^{\gamma,\tau}_{p,q_1,w}$ is monotone with respect to q, namely, for all $q_1, q_2 \in (0, \infty]$,

$$A^{\gamma,\tau}_{p,q_1,w} \hookrightarrow A^{\gamma+\epsilon,\tau}_{p,q_2,w}.$$

3. For all $p \in (0, \infty)$,

$$B^{\gamma,\tau}_{p,\min(p,q),w} \hookrightarrow F^{\gamma,\tau}_{p,q,w} \hookrightarrow B^{\gamma,\tau}_{p,\max(p,q),w}.$$

In particular, if $p \in (0, \infty)$, then $B^{\gamma,\tau}_{p,p,w} = F^{\gamma,\tau}_{p,p,w}$.

Definition 4.2. For $f \in S'$, let

$$||f||^1_{F^{\gamma,\tau}_{p,q,w}} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=0}^{\infty} 2^{j\gamma q}|(\nu_j f)(x)|^q\right)^{p/q} w(x)dx\right]^{1/p}$$

and

$$||f||^2_{F^{\gamma,\tau}_{p,q,w}} = \sup_{Q\in\mathcal{Q}} \sup_{x\in Q} |Q|^{-\gamma/n} w(Q)^{1/p-\tau} |\nu_{\nu_{jQ}} f(x)|.$$

Definition 4.3. For $f \in S'$, let

$$||f||^1_{B^{\gamma,\tau}_{p,q,w}} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\sum_{j=0}^{\infty} \left(\int_Q |2^{j\gamma}\nu_j f|^p w(x)\right)^{q/p} dx\right]^{1/q}$$

and

$$||f||^2_{B^{\gamma,\tau}_{p,q,w}} = \sup_{Q\in\mathcal{Q}} \sup_{x\in Q} |Q|^{-\gamma/n} w(Q)^{1/p-\tau} |\nu_{\nu_{jQ}} f(x)|.$$

Theorem 4.1. Let $f \in S'$.

1. If $\tau - 1/p < 0$, then $f \in A^{\gamma,\tau}_{p,q,w}$ if and only if $||f||^1_{A^{\gamma,\tau}_{p,q,w}} < \infty$. Moreover,

$$||f||_{A^{\gamma,\tau}_{p,q,w}} \equiv ||f||^1_{A^{\gamma,\tau}_{p,q,w}}.$$

2. If $\tau - 1/p > 0$, then $f \in A^{\gamma,\tau}_{p,q,w}$ if and only if $||f||^2_{F^{\gamma,\tau}_{p,q,w}} < \infty$. Moreover,

$$||f||_{A^{\gamma,\tau}_{p,q,w}} \equiv ||f||^2_{A^{\gamma,\tau}_{p,q,w}}.$$

In particular, if $\tau - 1/p > 0$, then $||f||_{B^{\gamma,\tau}_{p,q,w}} \equiv ||f||_{F^{\gamma,\tau}_{p,q,w}}$.

Proof. We consider only the Triebel case. To prove (1), it suffices to show that

$$||f||^1_{F^{\gamma,\tau}_{p,q,w}} \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}$$

since the inverse inequality obviously holds true by definitions. Put

$$I_Q = \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=0}^{\infty} 2^{j\gamma q}|\nu_j f(x)|^q\right)^{p/q} w(x)dx\right]^{1/p}.$$

Then

$$\begin{aligned} I_Q &= \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=0}^{j_Q\vee 0-1} 2^{j\gamma q}|\nu_j f(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &+ \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q\vee 0}^{\infty} 2^{j\gamma q}|\nu_j f(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \\ &\preceq J_Q + ||f||_{F^{\gamma,\tau}_{p,q,w}}, \end{aligned}$$

where $J_Q = \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=0}^{j_Q\vee 0-1} 2^{j\gamma q}|\nu_j f(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} = 0$ if $j_Q \le 0$. We only need to estimate J_Q in the case that $j_Q > 0$. For any $j \in \mathbb{N}$ with $j \le j_Q - 1$, there exists a unique dyadic cube Q_j such that $Q \subset Q_j$ and $l(Qj) = 2^{-j}$. Taking $x \in Q$ and $y \in Q_j$ then it is easy to see that

$$\nu_j f(x) \preceq \inf_{y\in Q_j} (\nu_j^\star f)_\lambda(y)$$

for all $\lambda > 0$. Then choosing $r \in (0, \min\{1, p, q\})$ to find

$$\begin{aligned}
&\frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=0}^{j_Q-1} 2^{j\gamma q}|\nu_j f(x)|^q\right)^{p/q} w(x)dx\right]^{1/p}\\
&\preceq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=0}^{j_Q-1} 2^{j\gamma r}|\nu_j f(x)|^r\right)^{p/r} w(x)dx\right]^{1/p}\\
&\preceq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=0}^{j_Q-1} 2^{j\gamma r}|\inf_{y\in Q_j} (\nu_j^\star f)_\lambda(y)|^r\right)^{p/r} w(x)dx\right]^{1/p}\\
&\preceq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=0}^{j_Q-1} \left(\frac{1}{w(Q_j)}\int_{Q_j} 2^{j\gamma p}(\nu_j^\star f)_\lambda(x)|^p w(x)\right)^{r/p}\right)^{p/r} w(x)dx\right]^{1/p}\\
&\preceq \left[\sum_{j=0}^{j_Q-1} \frac{[w(Q_j)]^{\tau r}}{[w(Q)]^{\tau r}}\frac{[w(Q)]^{r/p}}{[w(Q_j)]^{r/p}}\right]^{1/r} ||f||_{F^{\gamma,\tau}_{p,q,w}} = \left[\sum_{j=0}^{j_Q-1}\left(\frac{w(Q_j)}{w(Q)}\right)^{r\tau-r/p}\right]^{1/r} ||f||_{F^{\gamma,\tau}_{p,q,w}}\\
&\preceq \left[\sum_{j=0}^{j_Q-1} 2^{-(j-j_Q)r\delta n(\tau-1/p)}\right]^{1/r} ||f||_{F^{\gamma,\tau}_{p,q,w}} \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}, \quad \text{since } \tau - 1/p < 0.
\end{aligned}$$

Now we prove (2). We have for $x \in Q$

$$\begin{aligned}
&|Q|^{-\gamma/n} w(Q)^{1/p-\tau}|\nu_{j_Q} f(x)|\\
&\preceq w(Q)^{1/p-\tau} \inf_{y\in Q}(|Q|^{-\gamma/n}\nu_{j_Q}^\star f)_\lambda(y)\\
&\preceq w(Q)^{1/p-\tau} w(Q)^{-1/p+\tau}||f||_{F^{\gamma,\tau}_{p,q,w}} = ||f||_{F^{\gamma,\tau}_{p,q,w}}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\frac{1}{[w(Q)]^{\tau}}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^{\infty}|2^{j\gamma}\nu_j f(x)|^q\right)^{p/q}w(x)dx\right]^{1/p}\\
&\preceq \frac{1}{[w(Q)]^{\tau}}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^{\infty}\left(\sum_{l(P)=2^{-j},\,P\subset Q}|2^{j\gamma}\nu_j f(x)\chi_P(x)|\right)^q\right)^{p/q}w(x)dx\right]^{1/p}\\
&\preceq ||f||^2_{F^{\gamma,\tau}_{p,q,w}}\frac{1}{[w(Q)]^{\tau}}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^{\infty}\left(\sum_{l(P)=2^{-j},\,P\subset Q}w(P)^{\tau-1/p}\chi_P(x)\right)^q\right)^{p/q}w(x)dx\right]^{1/p}\\
&\preceq ||f||^2_{F^{\gamma,\tau}_{p,q,w}}\frac{1}{[w(Q)]^{\tau}}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^{\infty}\left(\sum_{l(P)=2^{-j},\,P\subset Q}w(P)^{\tau-1/p}\chi_P(x)\right)^q\right)^{p/q}w(x)dx\right]^{1/p}\\
&\preceq ||f||^2_{F^{\gamma,\tau}_{p,q,w}}\left[\left(\sum_{j=j_Q\vee 0}^{\infty}\left(\sum_{l(P)=2^{-j},\,P\subset Q}\left(\frac{w(P)}{w(Q)}\right)^{\tau-1/p}\chi_P(x)\right)^q\right)^{p/q}\right]^{1/p}\\
&\preceq ||f||^2_{F^{\gamma,\tau}_{p,q,w}}\left[\left(\sum_{j=j_Q\vee 0}^{\infty}2^{-n(\tau-1/p)(j-j_Q)\delta}\right)\right]^{1/p}\\
&\preceq ||f||^2_{F^{\gamma,\tau}_{p,q,w}}.
\end{aligned}
$$

□

As an application of Theorem 4.1, we prove that the space $F^{0,\tau}_{p,2,w}$ coincides with the Morrey space $M^{p,\tau}_w$, which is defined to be the set of all measurable functions f such that

$$||f||_{M^{p,\tau}_w}=\sup_{Q\in\mathcal{Q}}\frac{1}{[w(Q)]^{\tau}}\left[\int_Q|f|^p w(x)dx\right]^{1/p}.$$

Corollary 4.3. Let $w\in A_p, 1<p<\infty$ and assume that $\tau-1/p<0$. Then

$$M^{p,\tau}_w\equiv F^{0,0}_{p,2,w}$$

with equivalent norms.

Proof. We first prove that $M^{p,\tau}_w\hookrightarrow F^{0,0}_{p,2,w}$. By Theorem 4.1, it suffices to show that, for all $f\in M^{p,\tau}_w$

$$\sup_{Q\in\mathcal{Q}}\frac{1}{[w(Q)]^{\tau}}\left[\int_Q\left(\sum_{j=0}^{\infty}|\nu_j f(x)|^2\right)^{p/2}w(x)dx\right]^{1/p}\preceq||f||_{M^{p,\tau}_w}. \tag{4.8}$$

For a dyadic cube $Q=Q(x_0,r)$, set $f_1=f\chi_{Q(x_0,2r)}$ and $f_2=f-f_1$. Then it is well known that

$$I_1=\frac{1}{[w(Q)]^{\tau}}\left[\int_Q\left(\sum_{j=0}^{\infty}|\nu_j f_1(x)|^2\right)^{p/2}w(x)dx\right]^{1/p}\preceq\frac{1}{[w(Q)]^{\tau}}\left[\int_{\mathbb{R}^n}|f_1|^p w(x)dx\right]^{1/p},$$

see for instance Theorem 1.10 in [26]. This leads to $I_1 \preceq ||f||_{M_w^{p,\tau}}$. On the other hand, we have

$$I_2 = \left(\sum_{j=0}^{\infty} |\nu_j f_2(x)|^2\right)^{1/2} \preceq \sum_{k=1}^{\infty} \int_{S_k} \frac{|f(y)|}{|x-y|^n} dy \preceq \sum_{k=1}^{\infty} \int_{S_k} \frac{|f(y)|}{(2^k r + |x-y|)^n} dy$$

where, for $k \in \mathbb{N}, S_k = Q_{k+1} \setminus Q_k$ and $Q_k = Q(x_0, 2^k r)$. Using Holder's inequality, to get

$$\begin{aligned} I_2 &\preceq \sum_{k=1}^{\infty} \left(\int_{S_k} |f(y)|^p w(y) dy\right)^{1/p} \left(\int_{S_k} \frac{w^{-p'/p}(y)}{(2^k r + |x-y|)^{np'}} dy\right)^{1/p'} \\ &\preceq \sum_{k=1}^{\infty} \left(\int_{S_k} |f(y)|^p w(y) dy\right)^{1/p} |Q_{k+1}|^{-1} \left(\int_{Q_{k+1}} w^{-p'/p}(y) dy\right)^{1/p'} \\ &\preceq \sum_{k=1}^{\infty} \left(\int_{Q_{k+1}} |f(y)|^p w(y) dy\right)^{1/p} \left(\int_{Q_{k+1}} w(y) dy\right)^{-1/p} \\ &\preceq ||f||_{M_w^{p,\tau}} \sum_{k=1}^{\infty} [w(Q_{k+1})]^{\tau} \left(\int_{Q_{k+1}} w(y) dy\right)^{-1/p} = ||f||_{M_w^{p,\tau}} \sum_{k=1}^{\infty} [w(Q_{k+1})]^{\tau - 1/p}. \end{aligned}$$

It follows that,

$$\begin{aligned} &\frac{1}{[w(Q)]^{\tau}} \left[\int_Q \left(\sum_{j=0}^{\infty} |\nu_j f_2(x)|^2\right)^{p/2} w(x) dx\right]^{1/p} \preceq ||f||_{M_w^{p,\tau}} \sum_{k=1}^{\infty} \left[\frac{w(Q_{k+1})}{w(Q)}\right]^{\tau - 1/p} \\ &\preceq ||f||_{M_w^{p,\tau}} \sum_{k=1}^{\infty} \left(2^{nk\delta}\right)^{\tau - 1/p} \preceq ||f||_{M_w^{p,\tau}}. \end{aligned}$$

Similar argument leads to

$$F_{p,2,w}^{0,0} \hookrightarrow M_w^{p,\tau}.$$

□

Lemma 4.4. *Let $\tau \geq 0$ and $q \in (0, \infty]$ and $-\infty < \gamma_1 < \gamma_0 < +\infty$. Assume that $w \in A_\infty$ and for all $0 < t \leq 1$, $w(B(x,t)) \geq t^d$ for some $d > 0$ and all x. If*

$$0 < p_0 < p_1 \leq +\infty \text{ such that } \gamma_0 - \frac{d}{p_0} = \gamma_1 - \frac{d}{p_1},$$

then

$$B_{p_0,q,w}^{\gamma_0,\tau} \hookrightarrow B_{p_1,q,w}^{\gamma_1,\tau},$$

and if

$$0 < p_0 < p_1 < +\infty \text{ such that } \gamma_0 - \frac{d}{p_0} = \gamma_1 - \frac{d}{p_1},$$

then

$$F_{p_0,q,w}^{\gamma_0,\tau} \hookrightarrow F_{p_1,q,w}^{\gamma_1,\tau}.$$

Proof. We only prove the first inclusion. The proof of the second one is similar.

Let $m > 2n$ and let $B_J = B(x_0, 2^{-J})$ be any ball of $\mathbb{R}^n$ with radius 2^{-J}. Then for any $x \in B_J$ and for any $\nu \geq J \vee 0$ we have for $\phi_\nu f = f_\nu$

$$|f_\nu(x)| \preceq \left(\sum_{j=1}^{\infty} 2^{-jm} \int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^r (1 + |x-z| 2^\nu)^{-m} dz\right)^{1/r}.$$

Hence

$$
\begin{aligned}
|f_\nu(x)|^{p_1} &\preceq \left(\sum_{j=1}^{\infty} 2^{-jm} \int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^r (1+|x-z|2^\nu)^{-m} dz\right)^{p_1/r} \\
&\preceq \sum_{j=1}^{\infty} 2^{-jm} \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^r (1+|x-z|2^\nu)^{-m} dz\right)^{p_1/r} \\
&= \sum_{j=1}^{\infty} 2^{-jm} \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^r (1+|x-z|2^\nu)^{-m} dz\right)^{(p_1-p_0)/r} \\
&\times \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^r (1+|x-z|2^\nu)^{-m} dz\right)^{p_0/r} = \sum_{j=1}^{\infty} 2^{-im} I_{\nu m}(x) J_{\nu m}(x).
\end{aligned}
$$

To estimate $I(x)$ we use Holder's inequality and the B_p-condition to obtain,

$$
\begin{aligned}
&I_{\nu m}(x) \preceq \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0} \left(\int_{\mathbb{R}^n} 2^{\nu n} (1+|x-z|2^\nu)^{-mp_0/r} w^{(r-p_0)/p_0} dz\right)^{1-r/p_0} \\
&\preceq \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0} \left(\int_{\mathbb{R}^n} 2^{\nu n} (1+|x-z|2^\nu)^{-mp_0/r} w^{(r-p_0)/p_0} dz\right)^{(1-r/p_0)(p_1-p_0)/r} \\
&\preceq \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0} \left(\int_{|x-y|\le 2^{-\nu}} w^{(r-p_0)/p_0} dz\right)^{(1-r/p_0)(p_1-p_0)/r} \\
&\preceq \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0} \left(\int_{|x-y|\le 2^{-\nu}} w^{(r-p_0)/p_0} dz\right)^{(1-r/p_0)(p_1-p_0)/r} \\
&\preceq \left(\int_{B_{J-j-1}} 2^{\nu n} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0} \left(\int_{|x-y|\le 2^{-\nu}} w dz\right)^{(p_0-p_1)/p_0} 2^{n\nu(p_0-p_1)/p_0} \\
&\preceq 2^{d\nu(p_0-p_1)/p_0} \left(\int_{B_{J-j-1}} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0}.
\end{aligned}
$$

On the other hand

$$
J_{\nu m}(x) \preceq \left(M(|f_\nu|^r \chi_{B_{J-j-1}})\right)^{p_0/r}.
$$

We conclude that

$$
|f_\nu(x)|^{p_1} \preceq 2^{d\nu(p_0-p_1)/p_0} \sum_{j=1}^{\infty} 2^{-jm} \left(\int_{B_{J-j-1}} |f_\nu(z)|^{p_0} w(z) dz\right)^{(p_1-p_0)/p_0} \left(M(|f_\nu|^r \chi_{B_{J-j-1}})\right)^{p_0/r}.
$$

It follows by using Fefferman-Stein inequality that

$$
\int_{B_J} |f_\nu(x)|^{p_1} w(x) dx \preceq 2^{d\nu(p_0-p_1)/p_0} \sum_{j=1}^{\infty} 2^{-jm} \left(\int_{B_{J-j-1}} |f_\nu(z)|^{p_0} w(z) dz\right)^{p_1/p_0}.
$$

Now, since $p_0/p_1 \le 1$, we obtain

$$
\begin{aligned}
&\left(\int_{B_J} |f_\nu(x)|^{p_1} w(x) dx\right)^{1/p_1} \preceq 2^{d\nu(p_0-p_1)/(p_1 p_0)} \left[\sum_{j=1}^{\infty} 2^{-jm} \left(\int_{B_{J-j-1}} |f_\nu(z)|^{p_0} w(z) dz\right)^{p_1/p_0}\right]^{p_0/p_1 \cdot 1/p_0} \\
&\preceq 2^{d\nu(p_0-p_1)/(p_0 p_1)} \left[\sum_{j=1}^{\infty} 2^{-jmp_0/p_1} \int_{B_{J-j-1}} |f_\nu(z)|^{p_0} w(z) dz\right]^{1/p_0}.
\end{aligned}
$$

Thus if $\gamma_1 - d/p_1 = \gamma_0 - d/p_0$, then

$$\left(\int_{B_J} |2^{\gamma_1 \nu} f_\nu(x)|^{p_1} w(x)dx\right)^{q/p_1} \preceq \left[\sum_{j=1}^{\infty} 2^{-jm} \int_{B_{J-j-1}} |2^{\gamma_0 \nu} f_\nu(z)|^{p_0} w(z)dz\right]^{q/p_0}$$
$$\preceq \sum_{j=1}^{\infty} 2^{-jm\min(1,q/p_0)} \left[\int_{B_{J-j-1}} |2^{\gamma_0 \nu} f_\nu(z)|^{p_0} w(z)dz\right]^{q/p_0}.$$

This leads, by choosing m large enough, to

$$\sum_{\nu=J\vee 0}^{\infty} \left(\int_{B_J} |2^{\gamma_1 \nu} f_\nu(x)|^{p_1} w(x)dx\right)^{q/p_1} \preceq \sum_{j=1}^{\infty} 2^{-jm\min(1,q/p_0)} \sum_{\nu=(J-j-1)\vee 0}^{\infty} \left[\int_{B_{J-j-1}} |2^{\gamma_0 \nu} f_\nu(z)|^{p_0} w(z)dz\right]^{q/p_0}$$
$$\preceq w^{\tau q}(B_J) \sum_{j=1}^{\infty} 2^{-jm\min(1,q/p_0)}$$
$$\times \sum_{\nu=(J-j-1)\vee 0}^{\infty} \frac{w^{\tau q}(B_{J-j-1})}{w^{\tau q}(B_J)} \frac{1}{w^{\tau q}(B_{J-j-1})} \left[\int_{B_{J-j-1}} |2^{\gamma_0 \nu} f_\nu(z)|^{p_0} w(z)dz\right]^{q/p_0}$$
$$\preceq w^{\tau q}(B_J) \sum_{j=1}^{\infty} 2^{-jm\min(1,q/p_0)+\tau n p_0 j/r} ||f||^q_{B^{\gamma_0,\tau}_{p_0,q,w}} \preceq w^{\tau q}(B_J)||f||^q_{B^{\gamma_0,\tau}_{p_0,q,w}}.$$

□

5 Spaces on special Lipschitz domains

Definition 5.1. A special Lipschitz domain (SLD), $\Omega \subset \mathbb{R}^n$ is an open domain lying above the graph of a Lipschitz function $\theta : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, i.e,

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \theta(x')\},$$

and there is a positive constant A such that

$$|\theta(x') - \theta(y')| \leq A|x' - y'|, \ \ \forall x',\ y' \in \mathbb{R}^{n-1}.$$

Let $D(\Omega)$ be the set of all infinitely differentiable functions supported in Ω and denote by $D'(\Omega)$ its dual.

For a SLD Ω, the spaces $B^{\gamma,\tau}_{p,q,w}(\Omega)$ and $F^{\gamma,q}_{p,q,w}(\Omega)$ are defined to be the restriction of the corresponding spaces from $\mathbb{R}^n$ to Ω, i.e, we denote by $h|_\Omega = f$ in $D'(\Omega)$ the restriction of a distribution f on $\mathbb{R}^n$ to Ω:

$$A^{\gamma,\tau}_{p,q,w}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^{\gamma,q}_{p,w}, g|_\Omega = f\}$$

endowed with the quasi-norm:

$$||f||_{A^{\gamma,\tau}_{p,q,w}(\Omega)} = \inf\{||g||_{A^{\gamma,\tau}_{p,q,w}(\Omega)} : f = g|_\Omega, g \in A^{\gamma,\tau}_{p,q,w} = A^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)\}.$$

Consider a SLD $\Omega \subset \mathbb{R}^n$ as in Definition 5.1 and define the cone K with vertex at the origin by

$$K = \{(x', x_n) \in \mathbb{R}^n : |x'| \leq A^{-1}x_n\}.$$

Note that the cone K has the property that its shifts $x + K \subset \Omega$ for each $x \in \Omega$. Denote by $-K = \{-x : x \in K\}$ the reflected cone of K. Then for every $\nu \in D(-K)$ and $f \in D'(\Omega)$, the convolution $\nu \star f(x) = \langle f, \nu(x - \cdot)\rangle$ is well defined in Ω since $\operatorname{supp} \nu(x - \cdot) \subset \Omega$ for $x \in \Omega$.

Proposition 5.1. [25, Proposition 3.1] Let Ω be a special Lipschitz domain. The distribution $f \in D'(\Omega)$ belongs to $\mathcal{S}'(\Omega)$ if and only if there exists $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = g|_\Omega$.

Proposition 5.2. [25, Theorem 4.1 (a)] Let Ω be a special Lipschitz domain and K its associated cone. Then there exist functions $\phi_0, \phi, \psi_0, \psi \in \mathcal{S}$ supported in $-K$ such that

$$\int_{\mathbb{R}^n} x^\alpha \phi(x)dx = \int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0 \text{ for all multi-indices } \alpha, \tag{5.1}$$

and

$$f = \sum_{j=0}^{\infty} \psi_j \star \phi_j \star f \text{ in } D'(\Omega) \tag{5.2}$$

for all $f \in D'(\Omega)$.

Theorem 5.1 (Extension theorem)**.** Let $\phi_0, \phi, \psi_0, \psi \in \mathcal{S}$ be as in Proposition 5.2. Let $w \in A_\infty, 0 < p, q \leq \infty$ ($p < \infty$ in the $A^{\gamma,\tau}_{p,q,w}(\Omega)$-case). For any $g : \Omega \longrightarrow \mathbb{R}$ denote by g_Ω its extension from Ω to $\mathbb{R}^n$ by zero. Then the map $\mathcal{E}$ defined by

$$D'(\Omega) \ni f \stackrel{\mathcal{E}}{\longmapsto} \sum_{j=0}^{\infty} \psi_j \star (\phi_j \star f)_\Omega \text{ in } D'(\Omega) \tag{5.3}$$

yields a linear extension operator from $A^{\gamma,\tau}_{p,q,w}(\Omega)$ into the corresponding spaces on $\mathbb{R}^n$.

Remark 5.2. Theorem 5.1 is a generalization of Theorem 3.6 in [10] where w is assumed to be one. The key element of their proof is Lemma 3.3 with $w \equiv 1$. Since Lemma 3.3 is valid for an arbitrary $w \in A_\infty$, similar arguments in the proof of Theorem 3.6 in [10] lead to the results in Theorem 5.1.

Remark 5.3. Since in [10] the detailed proof is given only for Besov spaces, we provide the detailed proof for Triebel spaces for the sake of clarity.

Proof. Step 1. Let $(g_j)_{j\in\mathbb{N}_0}$ be a sequence of measurable functions and denote by $M^{g_j}_N(x)$, the Peetre maximal function of g_j, defined as

$$M^{g_j}_N(x) = \sup_{y\in\mathbb{R}^n} |g_j(y)|(1 + 2^j|x - y|)^{-N}, \quad N > 0.$$

We will prove that if $2^{j\gamma} M^{g_j}_N(x) \in L^\tau_{p,w}(l^q)$, then the series $\sum_{j=0}^\infty \psi_j \star g_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and $||\sum_{j=0}^\infty \psi_j \star g_j||_{F^{\gamma,\tau}_{p,q,w}} \preceq ||(2^{k\gamma}(M^{g_k}_N)_{k\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)}$.

We start with the following elementary inequality

$$\phi_k \star \psi_j \star g_j(x) \leq M^{g_j}_N(x) \int_{\mathbb{R}^n} |\psi_j \star \phi_k(y)|(1 + 2^j|y|)^N dy, \quad \forall M \geq 0. \tag{5.4}$$

Then we have the following Heideman type estimate, see [3]

$$\int_{\mathbb{R}^n} |\psi_j \star \phi_k(y)|(1 + 2^j|y|)^N \leq C_{M,N} 2^{-|k-j|M}, \quad \forall M \geq 0. \tag{5.5}$$

Taking $M > |\gamma| + nr_0\tau$ and put $\sigma = M - |\gamma|$, we obtain

$$2^{k\gamma}\phi_k \star \psi_j \star g_j(x) \preceq 2^{j\gamma}2^{-|k-j|\sigma}M_N^{g_j}(x). \tag{5.6}$$

Assume that

$$||(M_N^{g_j})_{n\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)} < \infty.$$

Then

$$\begin{aligned}
||\psi_j \star g_j||_{F^{\gamma-2\sigma,\tau}_{p,q,w}} &= \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{k=j_Q\vee 0}^{\infty} 2^{k(\gamma-2\sigma)q}|(\phi_k\star\psi_j\star g_j|^q\right)^{p/q} w(x)dx\right]^{1/p} \\
&\preceq 2^{-j\sigma}\sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{k=j_Q\vee 0}^{\infty} \left(2^{j\gamma}2^{-k\sigma-|k-j|\sigma}M_N^{g_j}(x)\right)^q\right)^{p/q} w(x)dx\right]^{1/p} \\
&\preceq 2^{-j\sigma}||(2^{k\gamma}(M_N^{g_k})_{k\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)},
\end{aligned}$$

using that $|k-j| \geq j-k$. Therefore $\sum_{j=0}^\infty \psi_j \star g_j$ converges in $F^{\gamma-2\sigma,\tau}_{p,q,w}$ and hence in $\mathcal{S}'(\mathbb{R}^n)$, since $F^{\gamma-2\sigma,\tau}_{p,q,w} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. On the other hand, we have $2^{k\gamma}|\phi_k \star \sum_{j=0}^\infty \psi_j \star g_j(x)| \preceq \sum_{j=0}^\infty 2^{-|k-j|\sigma}2^{j\gamma}M_N^{g_j}(x)$. Then since $\sigma > n\tau r_0$ we have by Lemma 3.3

$$||\sum_{j=0}^\infty \psi_j \star g_j||_{F^{\gamma,\tau}_{p,q,w}} \preceq ||(2^{k\gamma}(M_N^{g_k})_{k\in\mathbb{N}_0}||_{L^\tau_{p,w}(l^q)}. \tag{5.7}$$

Step 2. Let $f \in F^{\gamma,\tau}_{p,q,w}(\Omega)$. Then, for any $\epsilon > 0$, there exists $g \in F^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)$ such that $g|_\Omega = f$ in $D'(\Omega)$ and

$$||g||_{F^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)} \leq ||f||_{F^{\gamma,\tau}_{p,q,w}(\Omega)} + \epsilon. \tag{5.8}$$

On the other hand, we have $\nu_j f = \nu_j g$ if $y \in \Omega$ since $\operatorname{supp}\nu_j \subset -K$ and $y + K \subset \Omega$ for any point $y \in \Omega$. In consequence $\sup_{y\in\Omega}\frac{|\nu_j f(y)|}{(1+|x-y|)^\lambda} \preceq \sup_{y\in\mathbb{R}^n}\frac{|\nu_j f(y)|}{(1+|\bar{x}-y|)^\lambda}$, where $\bar{x} = (x', 2\theta(x') - x_n) \in \Omega$ is the symmetric point to $x = (x', x_n) \notin \bar{\Omega}$ with respect to $\partial\Omega$. Let $g_j = (\nu_j f)_\Omega$. Then by [25, p. 248] we have:

$$\sup_{y\in\Omega}\frac{|g_j(y)|}{(1+|x-y|)^\lambda}\begin{cases} = \sup_{y\in\Omega}\frac{|\nu_j f(y)|}{(1+|x-y|)^\lambda} & \text{if } x\in\Omega, \\ \preceq \sup_{y\in\Omega}\frac{|\nu_j f(y)|}{(1+|\bar{x}-y|)^\lambda}, & \text{if } x\notin\bar{\Omega}. \end{cases} \tag{5.9}$$

Using 5.7, 5.8, 5.9 and arguing as in the proof of Theorem 3.6 given in [10], we get

$$\begin{aligned}
&||\mathcal{E}f||_{F^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)} \leq ||\sum_{j=0}^\infty \psi_j \star (\phi_j\star f)_\Omega||_{F^{\gamma,\tau}_{p,q,w}(\Omega)} \preceq ||(2^{k\gamma}(M_N^{g_k})_{k\in\mathbb{N}_0}||_{l^q(L^\tau_{p,w})} \\
&\preceq \sup_{Q\in\mathcal{Q}}\frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{j=j_Q\vee 0}^\infty 2^{j\gamma q}|(\phi_j^\star h)_N(x)|^q\right)^{p/q} w(x)dx\right]^{1/p} \preceq ||h||_{F^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)} \\
&\preceq ||f|_{F^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)} + \epsilon.
\end{aligned}$$

Letting ϵ go to zero, we conclude that the extension operator is a bounded linear operator from $F^{\gamma,\tau}_{p,q,w}(\Omega)$ into $F^{\gamma,\tau}_{p,q,w}(\mathbb{R}^n)$. Finally, since the supports of ψ_0 and ψ lie in $-K$, it follows that

$$(\mathcal{E}f)_\Omega = \sum_{j=0}^\infty \psi_j\star\phi_j\star f = f \text{ in } D'(\Omega)$$

which completes the proof. □

6 Conclusion

This paper introduces a framework for non-homogeneous weighted Besov-type and Triebel-Lizorkin-type spaces by incorporating Muckenhoupt weights into the framework developed by Dachun Yang and Wen Yuan. The main contributions include local means characterizations, embedding theorems that unify several existing function spaces, and the construction of a Rychkov-type linear extension operator for these spaces.

We believe that the results presented in this paper can be naturally extended to weighted Besov-type and Triebel-Lizorkin-type spaces with variable exponents, opening new directions for future research.

Appendix

We adapt here the proof of Proposition 3.4 in [5].

Proof of Proposition 2.1. Set

$$||f||^B_{B^{\gamma,\tau}_{p,q,w}} \simeq \sup_{B_J} \frac{1}{[w(B_J)]^\tau} \left[\sum_{j=J\vee 0}^{\infty} \left(\int_{B_J} |2^{j\gamma} \nu_j f|^p w(x) \right)^{q/p} dx \right]^{1/q}.$$

Let Q be a dyadic cube. Then since $Q \subset B(x_Q, \sqrt{n} l(Q))$ and $|Q| \simeq |B(x_Q, \sqrt{n} l(Q))|$ we have by Lemma 2.1 $w(Q) \simeq w(B(x_Q, \sqrt{n} l(Q)))$. It follows

$$\begin{aligned} ||f||_{B^{\gamma,\tau}_{p,q,w}} &\preceq \sup_{Q \in \mathcal{Q}} \frac{1}{[w(B(x_Q, \sqrt{n} l(Q)))]^\tau} \left[\sum_{j=j_Q \vee 0}^{\infty} \left(\int_{B(x_Q, \sqrt{n} l(Q))} |2^{j\gamma} \nu_j f|^p w(x) \right)^{q/p} dx \right]^{1/q} \\ &\preceq ||f||^B_{B^{\gamma,\tau}_{p,q,w}}. \end{aligned}$$

On the other hand, by the properties of the dyadic cubes, there exists $k \in \mathbb{N}$ not depending on J such that

$$B_J \subset \cup_{\nu=1}^{k} Q^\nu_J, \quad B_J \cap Q^\nu_J \neq \emptyset, \quad \forall \nu = 1, \ldots, k.$$

Here, Q^ν_J is a dyadic cube of side length 2^{-J}. Now, since $Q^\nu_J \subset (2\sqrt{n} + 1) B_J$, Lemma 2.1 implies $w(Q^\nu_J) \preceq w(B_J)$. Hence,

$$\begin{aligned} &\sup_{B_J} \frac{1}{[w(B_J)]^\tau} \left[\sum_{j=J\vee 0}^{\infty} \left(\int_{B_J} |2^{j\gamma} \nu_j f|^p w(x) \right)^{q/p} dx \right]^{1/q} \\ &\preceq \sum_{\nu=1}^{k} \sup_{Q^\nu_J} \frac{1}{[w(Q^\nu_J)]^\tau} \left[\sum_{j=J\vee 0}^{\infty} \left(\int_{Q^\nu_J} |2^{j\gamma} \nu_j f|^p w(x) \right)^{q/p} dx \right]^{1/q} \\ &\preceq ||f||_{B^{\gamma,\tau}_{p,q,w}}. \end{aligned}$$

Now set

$$||f||^B_{F^{\gamma,\tau}_{p,q,w}} \simeq \sup_{B_J} \frac{1}{[w(B_J)]^\tau} \left[\int_{B_J} \left(\sum_{j=J\vee 0}^{\infty} 2^{j\gamma q} |\nu_j f|^q \right)^{p/q} w(x) dx \right]^{1/p}.$$

Then similar argument as before leads to

$$
\begin{aligned}
||f||_{F^{\gamma,\tau}_{p,q,w}} &\preceq \sup_{Q\in\mathcal{Q}} \frac{1}{[w(B(x_Q,\sqrt{n}l(Q)))]^{\tau}} \left[\int_{B(x_Q,\sqrt{n}l(Q))} \left(\sum_{j=J\vee 0}^{\infty} 2^{j\gamma q}|\nu_j f|^q\right)^{p/q} w(x)dx\right]^{1/p} \\
&\preceq ||f||^{B}_{F^{\gamma,\tau}_{p,q,w}} \preceq \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q^{\nu}_J)]^{\tau}} \left[\int_{Q^{\nu}_J} \left(\sum_{j=J\vee 0}^{\infty} 2^{j\gamma q}|\nu_j f|^q\right)^{p/q} w(x)dx\right]^{1/p} \preceq ||f||_{F^{\gamma,\tau}_{p,q,w}}.
\end{aligned}
$$

□

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