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Tridiagonal matrices representations of generalized bivariate complex polynomials

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Abstract. This paper introduces a closed-form expression for generalized bivariate complex polynomials using a tridiagonal determinant in symbolic form. Subsequences with even and odd indices are also represented as tridiagonal determinants. Furthermore, we achieve factorizations of the tridiagonal determinant in terms of eigenvalues and demonstrate the versatility of the established closed-form formula through its application in deriving Chebyshev polynomials. This generalized sequence encompasses several well-known polynomial families in one or two variables, including the Fibonacci, Lucas, Pell, PellLucas, and Chebyshev polynomial sequences in tridiagonal representation. Finally, we explore identities involving complex bivariate generalized polynomials, expressed as second-order determinants.

Keywords: Matrices, Bivariate, Complex Polynomials, Recurrence, Symbolic, Summation.

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1 Introduction

The relationship between matrix theory and integer sequences (see [7,10,18–22]) has long captivated mathematicians. Among these sequences, Fibonacci numbers stand out due to their rich combinatorial properties and wide-ranging applications in number theory, algebra, and computer science. An elegant approach to studying Fibonacci numbers and their generalizations involves representing them through tridiagonal matrices, whose determinants often produce Fibonacci numbers, Lucas numbers, or related polynomial forms.

Tridiagonal matrices, characterized by nonzero entries along the main diagonal and the two adjacent diagonals, frequently appear in numerical analysis, finite difference methods, and physical modeling. Beyond their practical applications, these matrices exhibit intriguing algebraic properties. When their entries are chosen appropriately, the determinants follow recursive patterns similar to those of Fibonacci sequences. This observation has inspired extensive research connecting the determinants of structured matrices to classical integer sequences.

Researchers such as [8], and more definitively [1–4, 10, 18, 21, 23], have demonstrated that the determinants of certain classes of tridiagonal matrices directly produce Fibonacci numbers, offering a compelling matrix-based characterization of these sequences. The concept

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of Fibonacci polynomials broadens this framework, where matrix entries depend on parameters or variables, resulting in polynomial generalizations of Fibonacci numbers. Scholars like [?, 5, 6, 11, 12, 14, 16] and [15] have made significant contributions by deriving explicit determinant formulas for these generalized Fibonacci polynomials through tridiagonal matrices and developing efficient algorithms for their computation.

Simultaneously, Fibonacci polynomials provide a versatile framework for analyzing generalized recurrence relations, with the classical Fibonacci sequence emerging as a special case. These polynomials preserve the recursive structure of Fibonacci numbers while incorporating additional information through variable coefficients, making them valuable tools in approximation theory and symbolic computation.

This paper investigates the determinant structure of tridiagonal matrices and their relationship to Fibonacci numbers and polynomials by analyzing the algebraic properties of such matrices to derive explicit formulas for their determinants and explore how they naturally generate Fibonacci sequences and their polynomial analogues. The study is organized as follows: Section 1 provides a literature survey of the field; Section 2 defines the generalized bivariate complex second-order recurrence, polynomials, generating functions, and the Binet formula; Section 3 introduces the tridiagonal matrices of generalized bivariate complex polynomials, highlighting the algebraic harmony between linear recurrence relations and matrix theory; Section 4 presents the tridiagonal matrices for subsequences of even and odd indices; Section 5 considers the factorization of complex bivariate polynomials; and Section 6 discusses complex bivariate generalized polynomial identities expressed as second-order determinants. This unified framework generalizes several well-known sequences, including the Fibonacci, Lucas, Pell, and PellLucas sequences.

2 Generalized bivariate complex polynomials

Definition 2.1. [23] If $K_1(a_1,a_2) = a_1x + ia_2y$, $K_2(b_1,b_2) = b_1x + ib_2y$, $p(p_1,p_2) = p_1x + ip_2y$, and $q(q_1,q_2) = q_1x + iq_2y$, $i = \sqrt{-1}$, a_j, b_j, p_j, q_j, $(j = 1,2)$ are real numbers, $q \neq 0$ and $K_2 \neq 0$, we define a generalized bivariate complex second order recurrence

$$K_n(p,q,K_1,K_2) = pK_{n-1}(p,q,K_1,K_2) + iqK_{n-2}(p,q,K_1,K_2),\ n \geq 3. \tag{2.1}$$

2.1 Some initial terms of bivariate complex polynomial sequences in symbolic form

$$K_n(p,q,K_1,K_2) = \left\{\begin{array}{l} a_1x + ia_2y, b_1x + ib_2y, (p_1x + ip_2y)(b_1x + ib_2y) + i(q_1x + iq_2y)(a_1x + ia_2y), \\ \left[(p_1x + ip_2y)^2 + i(q_1x + iq_2y)\right](b_1x + ib_2y) \\ + i(p_1x + ip_2y)(q_1x + iq_2y)(a_1x + ia_2y), \\ \left[(p_1x + ip_2y)^3 + 2i(p_1x + ip_2y)(q_1x + iq_2y)\right](b_1x + ib_2y) \\ + \left[i(p_1x + ip_2y)^2(q_1x + iq_2y) - (q_1x + iq_2y)^2\right](a_1x + ia_2y), \\ \left[(p_1x + ip_2y)^4 + 3i(p_1x + ip_2y)^2(q_1x + iq_2y) - (q_1x + iq_2y)^2\right](b_1x + ib_2y) \\ + \left[i(p_1x + ip_2y)^3(q_1x + iq_2y) - 2(p_1x + ip_2y)(q_1x + iq_2y)^2\right](a_1x + ia_2y), \cdots \end{array}\right\}. \tag{2.2}$$

2.2 Generating function

Theorem 2.2. *See [23]. If function $G(t)$is the generating function of Generalized complex polynomial sequences of two variables defined in (1) ,then*

$$G(t) = \frac{K_1 + tK_2}{1 - pt - iqt^2}. \tag{2.3}$$

In expanded form

$$G(t) = \frac{(a_1x + ia_2y) + (b_1x + ib_2y)\, t}{1 - (p_1x + ip_2y)\, t - i\,(q_1x + iq_2y)\, t^2}.$$

2.3 Binet formula

Theorem 2.3. *See [23].*

$$K_n(\alpha,\beta,K_j) = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) K_2 - \left(\frac{\alpha^n\beta - \beta^n\alpha}{\alpha - \beta}\right) K_1 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) K_2 - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \alpha\beta K_1. \tag{2.4}$$

In expanded form

$$K_n\,(\alpha,\beta,K_1,K_2) = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)(b_1x + ib_2y) - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)\alpha\beta\,(a_1x + ia_2y)\,,$$

where $\alpha,\beta = \left(p \pm \sqrt{p^2 + 4iq}\right)/2$ *and* $\alpha + \beta = p, \alpha\beta = -iq$*, and* $\alpha - \beta = \sqrt{p^2 + 4iq}$*.*

For the sake of minimalism, we will denote K_n, K_1, K_2, p and q in place of $K_n(p,q,K_1,K_2), K_1(a_1a_2)$, $K_2(b_1,b_2), p(p_1,p_2)$, and $q(q_1,q_2)$respectively, throughout the remainder of this paper. Further K_n, K_1, K_2, p and q are functions of x and y.

3 Tridiagonal matrices of generalized bivariate complex polynomials

A general $n \times n$ tridiagonal matrix is a special kind of square matrix that has nonzero elements only on the main diagonal, superdiagonal subdiagonal. All other elements are zero. A General form of a tridiagonal matrix M_n is

$$M_n = \begin{pmatrix} m_{1,1} & m_{1,2} & 0 & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & 0 & \cdots & 0 \\ 0 & m_{3,2} & m_{3,3} & m_{3,4} & \cdots & 0 \\ 0 & 0 & m_{4,3} & m_{4,4} & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & m_{n-1,n} \\ 0 & 0 & 0 & 0 & m_{n,n-1} & m_{n,n} \end{pmatrix}.$$

The determinants of M_n are given by recursive formula

$$\det M_n = a_{n,n} \det M_{n-1} - a_{n,n-1}a_{n-1,n} \det M_{n-2}.$$

3.1 Tridiagonal representation of family of matrices for generalized bivariate complex polynomials

Let $\{M_n,\ n = 1,2,3\cdots\}$ be a sequence of complex tridiagonal matrices in the form

$$M_n(p,q,K_1,K_2) = (m_{jk})_{n\times n} = \begin{cases} K_2, \text{ if } j=k,\ j=\ k=1 \\ p, \text{ if } j=k,\ 2\le j,k\le n \\ -q, \text{ if } k-j=1,\ 1\le j\le n-1,\ 2\le k\le n \\ iK_1, \text{ if } j-k=1,\ j=2,k=1 \\ i, \text{ if } j-k=1,\ 3\le j\le n, 2\le k\le n-1 \\ 0 \text{ , otherwise} \end{cases}$$

such that

$$M_n = \begin{pmatrix} K_2 & -q & 0 & 0 & \cdots & 0 \\ iK_1 & p & -q & 0 & \cdots & 0 \\ 0 & i & p & -q & \cdots & 0 \\ 0 & 0 & i & p & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -q \\ 0 & 0 & 0 & 0 & i & p \end{pmatrix},$$

where K_1, K_2, p, q are defined in (2.1). Then, it is easy to see that that the determinant of M_n i.e.

$$\det M_n = K_{n+1},$$

is the generalized complex bivariate polynomials.

Theorem 3.1. *For the tridiagonal matrix M_n reprsenting the generalized bivariate complex polynomials*

$$M_n = \begin{pmatrix} K_2 & -q & 0 & 0 & \cdots & 0 \\ iK_1 & p & -q & 0 & \cdots & 0 \\ 0 & i & p & -q & \cdots & 0 \\ 0 & 0 & i & p & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -q \\ 0 & 0 & 0 & 0 & i & p \end{pmatrix} = K_{n+1}\left(p,q,K_1,K_2\right).$$

Proof. For $n = 1,2,3\cdots$,we have
If $n = 1$,

$$M_1 = (K_2)_{1\times 1}.$$

This implies $\det(M_1) = K_2$.
If $n = 2$,

$$M_2 = \begin{pmatrix} K_2 & -q \\ iK_1 & p \end{pmatrix}_{2\times 2}.$$

This implies $\det\,(M_2) = pK_2 + iqK_1 = K_3(p,q,K_1,K_2)$.
If $n = 3$,

$$M_3 = \begin{pmatrix} K_2 & -q & 0 \\ iK_1 & p & -q \\ 0 & i & p \end{pmatrix},$$

this implies

$$\begin{aligned}\det(M_3) &= \left(p^2+iq\right)K_2+ipqK_1\\ &= p\left(pK_2+iqK_1\right)+iqK_2\\ &= pK_3+iqK_2 = K_4(p,q,K_1,K_2).\end{aligned}$$

Assume that the theorem is true for $(n-1)$ and $(n-2)$ then

$$\det M_{n-1} = K_n(p,q,K_1,K_2)$$

and

$$\det M_{n-2} = K_{n-1}(p,q,K_1,K_2)$$

Since

$$\det M_n = m_{n,n}\det M_{n-1} - m_{n,n-1}m_{n-1,n}\det M_{n-2},$$

$$\begin{aligned}\det M_n &= p\det K_n(p,q,K_1,K_2) - i\,(-q)\det K_{n-1}(p,q,K_1,K_2)\\ \det M_n &= p\det K_n(p,q,K_1,K_2) + iq\det K_{n-1}(p,q,K_1,K_2) = K_{n+1}(p,q,K_1,K_2)^{\cdot}\end{aligned}$$

□

Hence the result.

4 Subsequences of complex bivariate generalized polynomials

Definition 4.1.

$$E_n(p,q,K_1,K_2) = \left(e_{jk}\right)_{n\times n} = \begin{cases}\left(p^2+iq\right)K_2+ipqK_1, \text{ if } j=k,\ j=\ k=1\\ 2iq+p^2, \text{ if } j=k,\ 2\le j,k\le n\\ iq^2, \text{ if } k-j=1,\ 1\le j\le n-1,\ 2\le k\le n\\ iK_2, \text{ if } j-k=1,\ j=2,k=1\\ i, \text{ if } j-k=1,\ 3\le j\le n, 2\le k\le n-1\\ 0 \text{ , otherwise}\end{cases} \tag{4.1}$$

4.1 Sequences of even indices of complex bivariate generalized polynomials

$$E_n(p,q,K_1,K_2) = \begin{pmatrix}\left(p^2+iq\right)K_2+ipqK_1 & iq^2 & 0 & 0 & \cdots & 0\\ iK_2 & 2iq+p^2 & iq^2 & 0 & \cdots & 0\\ 0 & i & 2iq+p^2 & iq^2 & \cdots & 0\\ 0 & 0 & i & 2iq+p^2 & \cdots & 0\\ \vdots & \vdots & 0 & \vdots & \ddots & iq^2\\ 0 & 0 & 0 & 0 & i & 2iq+p^2\end{pmatrix}$$

4.2 Sequences of odd indices of complex bivariate generalized polynomials

Definition 4.2.

$$O_n(p,q,K_1,K_2) = \left(e_{jk}\right)_{n\times n} = \begin{cases} pK_2 + iqK_1, \text{ if } j=k,\ j=\ k=1 \\ 2iq+p^2, \text{ if } j=k,\ 2\le j,k\le n \\ iq^2, \text{ if } k-j=1,\ 1\le j\le n-1,\ 2\le k\le n \\ iK_1, \text{ if } j-k=1,\ j=2, k=1 \\ i, \text{ if } j-k=1,\ 3\le j\le n, 2\le k\le n-1 \\ 0 \text{ , otherwise} \end{cases} \tag{4.2}$$

$$O_n(p,q,K_1,K_2) = \begin{pmatrix} pK_2+iqK_1 & iq^2 & 0 & 0 & \cdots & 0 \\ iK_1 & 2iq+p^2 & iq^2 & 0 & \cdots & 0 \\ 0 & i & 2iq+p^2 & iq^2 & \cdots & 0 \\ 0 & 0 & i & 2iq+p^2 & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & iq^2 \\ 0 & 0 & 0 & 0 & i & 2iq+p^2 \end{pmatrix}_{n\times n}.$$

$n \ge 2$

5 Factorization of complex bivariate generalized polynomials

In this section generalization is considered for complex bivariate generalized polynomials, whereas [2] studied factorizations of Lucas and Fibonacci numbers.

Define sequence of matrices through another tridiagonal matrices by

Definition 5.1.

$$F_n(p,q,K_1,K_2) = \left(f_{jk}\right)_{n\times n} = \begin{cases} -iK_2 + i, \text{ if } j=k,\ j=\ k=1 \\ -ip+i, \text{ if } j=k,\ 2\le j,k\le n \\ iq, \text{ if } k-j=1,\ 1\le j\le n-1,\ 2\le k\le n \\ K_1, \text{ if } j-k=1,\ j=2, k=1 \\ 1, \text{ if } j-k=1,\ 3\le j\le n, 2\le k\le n-1 \\ 0 \text{ , otherwise} \end{cases} \tag{5.1}$$

Therefore, we have

$$A_n = \begin{pmatrix} -iK_2+i & iq & 0 & 0 & \cdots & 0 \\ K_1 & -ip+i & iq & 0 & \cdots & 0 \\ 0 & 1 & -ip+i & iq & \cdots & 0 \\ 0 & 0 & 1 & -ip+i & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & iq \\ 0 & 0 & 0 & 0 & 1 & -ip+i \end{pmatrix} \tag{5.2}$$

$$I + iA_n = \begin{pmatrix} K_2 & -q & 0 & 0 & \cdots & 0 \\ iK_1 & p & -q & 0 & \cdots & 0 \\ 0 & i & p & -q & \cdots & 0 \\ 0 & 0 & i & p & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -q \\ 0 & 0 & 0 & 0 & i & p \end{pmatrix}. \tag{5.3}$$

Evidently,

$$M_n = I + iA_n,$$

where I is an identity matrix of order n.

Let $\lambda_k, k = 1, 2, \cdots n$, be the eigenvalues of A_n (with associated eigenvectors v_k). Then, for each k, we have

$$\begin{aligned} M_n v_k &= [I + iA_n]\, v_k, \\ &= Iv_k + i\lambda_k v_k = v_k + i\lambda_k v_k = (1 + i\lambda_k)\, v_k. \end{aligned}$$

Therefore

$$e_k = 1 + i\lambda_k, k = 1, 2 \cdots n,$$

are the eigenvalues of M_n. Hence,

$$|M(n)| = \prod_{k=1}^{n} (1 + i\lambda_k), n \geq 1.$$

In order to determine the λ_k, we recall that each λ_k is a zero of the characteristic polynomial $f_n(\lambda) = |A_n - \lambda I|$. Since $A_n - \lambda I$ is a tridiagonal matrix, i.e.

$$A_n - \lambda I = \begin{pmatrix} -iK_2 + i - \lambda & iq & 0 & 0 & \cdots & 0 \\ K_1 & -ip + i - \lambda & iq & 0 & \cdots & 0 \\ 0 & 1 & -ip + i - \lambda & iq & \cdots & 0 \\ 0 & 0 & 1 & -ip + i - \lambda & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & iq \\ 0 & 0 & 0 & 0 & 1 & -ip + i - \lambda \end{pmatrix}.$$

Now, to establish a recursive formula for the characteristic polynomials of $\{A_n, n = 1, 2 \cdots\}$

$$\begin{aligned} &f_1(\lambda) = |A_1 - \lambda I| = -\lambda - iK_2 + i, \\ &f_2(\lambda) = |A_2 - \lambda I| = \lambda^2 + (iK_2 - 2i + ip)\,\lambda - (p-1)(K_2 - 1) - iqK_1, \\ &\cdots \quad \cdots \quad \cdots \\ &f_n(\lambda) = |A_3 - \lambda I| = (-ip + i - \lambda)\, f_{n-1}(\lambda) - iqf_{n-2}(\lambda). \end{aligned}$$

By applying the appropriate transformation, the above sequence of characteristic polynomials can be converted into a different sequence of characteristic polynomials ((see [13,17])

If $\lambda = -2x$, then

$$\begin{aligned} &S_1(x) = 2x - iK_2 + i, \\ &S_2(x) = 4x^2 - 2i(K_2 - 2 + p)\, x - (p-1)(K_2 - 1) - iqK_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &S_n(x) = (-ip + i + 2x)\, S_{n-1}(x) - iqS_{n-2}(x). \end{aligned}$$

The sequences of polynomials $\{S_n(x), n \geq 1\}$ reduces to is the set of polynomials $\{U_n(x), n \geq 1\}$ which is the set of type of Chebyshev polynomials of the second kind. In (2.1) setting

$$p_1 = \frac{1}{x}, p_2 = 0, q_1 = \frac{1}{ix}, q_2 = 0, a_1 = \frac{1}{x}, a_2 = 0, b_1 = \frac{1}{x}, b_2 = 0,$$

that is

$$p = 1, q = \frac{1}{i}, K_2 = 1, K_1 = 1.$$

We obtain

$$\begin{aligned} &U_1(x) = 2x, \\ &U_2(x) = 4x^2 - 1, \\ &U_3(x) = 8x^3 - 4x, \\ &\cdots, \\ &U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \;. \end{aligned}$$

We obtain the factorization of M_n for complex Fibonacci numbers as in equation (2.9) to (2.11) of [2]. **Chebyshev polynomials of the second kind** as being generated by determinants of successive tridiagonal matrices obtained from the general matrix M_n

$$p_1 = 2, p_2 = 0, q_1 = \frac{i}{x}, q_2 = 0, a_1 = 0, a_2 = 0, b_1 = \frac{1}{x}, b_2 = 0,$$

that is

$$p = 2x,\ q = i,\ K_1 = 0,\ K_2 = 1.$$

$$M_n(x) = \begin{pmatrix} 1 & -i & 0 & 0 & \cdots & 0 \\ 0 & 2x & -i & 0 & \cdots & 0 \\ 0 & i & 2x & -i & \cdots & 0 \\ 0 & 0 & i & 2x & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -i \\ 0 & 0 & 0 & 0 & i & 2x \end{pmatrix}.$$

Chebyshev polynomials of the first kind as being generated by determinants of successive tridiagonal matrices obtained from the general matrix M_n

$$p_1 = 2, p_2 = 0, q_1 = \frac{i}{x}, q_2 = 0, a_1 = \frac{1}{x}, a_2 = 0, b_1 = 1, b_2 = 0,$$

that is

$$p = 2x,\ q = i,\ K_1 = 1,\ K_2 = x.$$

$$M_n(x) = \begin{pmatrix} x & -i & 0 & 0 & \cdots & 0 \\ i & 2x & -i & 0 & \cdots & 0 \\ 0 & i & 2x & -i & \cdots & 0 \\ 0 & 0 & i & 2x & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -i \\ 0 & 0 & 0 & 0 & i & 2x \end{pmatrix}.$$

Chebyshev polynomials of the third kind as being generated by determinants of successive tridiagonal matrices obtained from the general matrix M_n

$$p_1 = 2, p_2 = 0, q_1 = \frac{i}{x}, q_2 = 0, a_1 = \frac{1}{x}, a_2 = 0, b_1 = 2 - \frac{1}{x}, b_2 = 0,$$

that is

$$p = 2x,\ q = i,\ K_1 = 1,\ K_2 = 2x - 1.$$

$$M_n(x) = \begin{pmatrix} 2x-1 & -i & 0 & 0 & \cdots & 0 \\ i & 2x & -i & 0 & \cdots & 0 \\ 0 & i & 2x & -i & \cdots & 0 \\ 0 & 0 & i & 2x & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -i \\ 0 & 0 & 0 & 0 & i & 2x \end{pmatrix}.$$

Chebyshev polynomials of the fourth kind as being generated by determinants of successive tridiagonal matrices obtained from the general matrix M_n

$$p_1 = 2, p_2 = 0, q_1 = \frac{i}{x}, q_2 = 0, a_1 = \frac{1}{x}, a_2 = 0, b_1 = 2 - \frac{1}{x}, b_2 = 0,$$

that is

$$p = 2x,\ q = i,\ K_1 = 1,\ K_2 = 2x + 1.$$

$$M_n(x) = \begin{pmatrix} 2x+1 & -i & 0 & 0 & \cdots & 0 \\ i & 2x & -i & 0 & \cdots & 0 \\ 0 & i & 2x & -i & \cdots & 0 \\ 0 & 0 & i & 2x & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -i \\ 0 & 0 & 0 & 0 & i & 2x \end{pmatrix}.$$

6 Complex bivariate generalized polynomials identities and second order determinants

On the basis of determinants simple proofs of the identities [5].

Theorem 6.1. *For $n \geq k$,*

$$K_{n+m+1}(p, q, K_1, K_2) = K_{n+1}K_{m+1}(p, q, 0, 1) + iqK_nK_m(p, q, 0, 1).$$

Proof. First replace k by $m+1$ and then n by $n-k$ to obtain the above identity as

$$K_n = K_{n-k+1}K_k(p, q, 0, 1) + iqK_{n-k}K_{k-1}(p, q, 0, 1).$$

Now for $k = 1, 2, 3 \cdots n$, we have

$$K_n = K_nK_1(p, q, 0, 1) + iqK_{n-1}K_0(p, q, 0, 1),$$

$$K_n = K_{n-1}K_2(p, q, 0, 1) + iqK_{n-2}K_1(p, q, 0, 1),\ ,$$

$$K_n = K_{n-2}K_3(p, q, 0, 1) + iqK_{n-3}K_2(p, q, 0, 1),\ ,$$

$$\cdots$$

$$K_n = K_{n-2}K_3(p, q, 0, 1) + iqK_{n-3}K_2(p, q, 0, 1),$$

$$K_n = K_2K_{n-1}(p, q, 0, 1) + iqK_1K_{n-2}(p, q, 0, 1),$$

$$K_n = K_1K_n(p, q, 0, 1) + iqK_0K_{n-1}(p, q, 0, 1).$$

Now consider a second order matrix Q

$$Q_k = \begin{bmatrix} K_{n-k}(p,q,0,1 & -K_k \\ K_{n-k-1}(p,q,0,1 & K_{k+1} \end{bmatrix}.$$

$$Q_0 = \begin{bmatrix} K_n(p,q,0,1) & -iqK_0 \\ K_{n-1}(p,q,0,1) & K_1 \end{bmatrix}, |Q_0| = K_1K_n(p,q,0,1) + iqK_0K_{n-1}(p,q,0,1) = K_n.$$

$$Q_1 = \begin{bmatrix} K_{n-1}(p,q,0,1 & -K_1 \\ K_{n-2}(p,q,0,1 & K_2 \end{bmatrix}, |Q_1| = K_2K_{n-1}(p,q,0,1) + iqK_1K_{n-2}(p,q,0,1) = K_n.$$

$$Q_2 = \begin{bmatrix} K_{n-2}(p,q,0,1 & -K_2 \\ K_{n-3}(p,q,0,1 & K_3 \end{bmatrix}, |Q_2| = K_3K_{n-2}(p,q,0,1) + iqK_2K_{n-3}(p,q,0,1) = K_n.$$

$$\cdots$$

$$Q_{k-1} = \begin{bmatrix} K_{n-k+1}(p,q,0,1 & -K_{k-1} \\ K_{n-k}(p,q,0,1 & K_k \end{bmatrix}, |Q_{k-1}| = K_kK_{n-k+1}(p,q,0,1) + iqK_{k-1}K_{n-k}(p,q,0,1) = K_n.$$

$$\cdots$$

This establish the identity.

□

7 Conclusion

In this paper, we present a unified closed-form expression for generalized bivariate complex polynomials via a symbolic tridiagonal determinant representation. By deriving subsequences with even and odd indices and establishing eigenvalue-based factorizations, we elucidate the structural richness of these polynomial families. We validate the proposed closed-form by recovering Chebyshev polynomials, thereby demonstrating its broad applicability. This generalization encapsulates numerous classical sequencesincluding Horadam, Fibonacci, Lucas, Pell, PellLucas, and Chebyshevwithin a coherent tridiagonal framework. Furthermore, we derive identities involving these polynomials using second-order determinants, offering deeper algebraic and combinatorial insights. This framework provides promising directions for future research, particularly in solving differential equations, spectral analysis, and quantum mechanics, where orthogonal polynomials and tridiagonal structures are foundational.

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