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An L^2-stability analysis of a $\theta-$scheme for a class of nonlinear parabolic variational inequalities of obstacle type

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Abstract. This paper analyzes the stability of a fully discrete finite element approximation for a class of nonlinear parabolic variational inequalities of obstacle type. The temporal discretization is based on a $\theta-$scheme. We derive a stability condition for the scheme that depends critically on the parameter θ. We prove that the method is unconditionally stable in the L^2-norm for $\theta \in \left[\frac{1}{2}, 1\right]$. For $\theta \in \left[0, \frac{1}{2}\right)$, we establish a precise Courant–Friedrichs–Lewy (CFL)-type condition, $\Delta t \leq \dfrac{2}{L\left(1-\theta\left(1-\theta\right)\right)}$, where L is the Lipschitz constant of the nonlinear source term. The analysis is based on a careful choice of test functions in the variational inequality and by deriving sharp estimates of the associated bilinear form.

Keywords: Parabolic variational inequality, obstacle problem, finite element method, $\theta-$scheme, stability analysis.

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1 Introduction

Parabolic variational inequalities (PVIs) naturally arise in many fields, including finance (for example, American option pricing, Bensoussan and Lions [3], Madi *et al.* [25]), mechanics (such as contact and unilateral problems, Brézis and Stampacchia [12], Duvaut and Lions [15]), and physics (e.g., free boundary and diffusion phenomena, Lions and Stampacchia [24], Rodrigues [27]). Among these, obstacle-type problems form a fundamental and extensively studied subclass, where the solution is constrained to lie above or below a given function, the "obstacle".

The elliptic obstacle problem, modeling the equilibrium position of an elastic membrane constrained to lie above a fixed obstacle, is a cornerstone of the theory. Its analysis dates back to the seminal works of Brézis and Stampacchia [12], Lions and Stampacchia [24], and Stampacchia [28], which laid the foundation for the mathematical treatment of variational inequalities. The parabolic counterpart, which is the focus of this paper, introduces the time dimension, making it suitable for modeling evolving systems like American options with early exercise features or the dynamics of contact problems.

PVIs also have important applications in engineering, for example in structural optimisation and fracture modelling (Glowinski [18]), in control theory for optimal stopping and constrained stochastic control problems (Bensoussan and Lions [3]). These problems are numerically challenging due to their inherent nonlinearity and the strong coupling between the differential operator and the inequality constraint (Kacur and Van Keer [22]).

Research on numerical methods for PVIs remains active and extensive. Semi-discrete methods, where time remains continuous while space is discretized (method of lines), are classical. Fully discrete methods usually rely on implicit time-stepping combined with techniques such as projection (Friedman [17]) or penalisation (Falk [16], Vuik [29]) to enforce constraints. More recently, innovative numerical frameworks have emerged, including space-time finite element methods that improve convergence and adaptivity (Boulbrachene [8–10], Boulbrachene *et al.* [7,11], Cortey-Dumont [14], Carvajal *et al.* [13]), and advanced adaptive finite element methods targeting obstacle problems (Gustafsson [19], Nochetto *et al.* [26]). Moreover, recent studies have provided generalized error estimates for parabolic variational inequalities (Alnashri [1], Bencheikh Le Hocine *et al.* [2], Boulaaras *et al.* [5,6], Haiour and Hadidi [20]) and explored new classes of problems involving hemivariational and degenerate parabolic operators (Han and Wang [21], Li and Bi [23]). These advances highlight the ongoing development aiming to better understand and efficiently approximate PVIs.

This paper contributes to this body of work by providing a precise L^2- stability analysis of a $\theta-$scheme applied to a nonlinear parabolic variational inequality of obstacle type. The stability condition we derive explicitly links the choice of the time-stepping parameter θ, the time step Δt, and the physical parameters of the problem (Lipschitz and coercivity constants), offering practical guidance for robust simulations.

The rest of this paper is organized as follows. In Section 2, we state the continuous problem, its assumptions, and its variational formulation. Section 3 introduces the semi-discrete and fully discrete finite element approximations. The core of our stability analysis for the $\theta-$scheme is detailed in Section 4. Finally, we conclude in Section 5.

2 The continuous problem

2.1 Preliminaries and notation

Let Ω be an open, bounded, and convex domain in $\mathbb{R}^d$ $(d \geq 1)$ with a smooth boundary denoted by $\partial\Omega$. We use standard notations for Lebesgue spaces L^p $(p \in [1, +\infty))$ and Sobolev spaces $W^{m,p}(\Omega)$. We denote $H^m(\Omega) = W^{m,2}(\Omega)$ and define $H_0^1(\Omega)$ the closure of the space of test functions $C_0^\infty(\Omega)$ with respect to the $\|\cdot\|_{H^1(\Omega)}$ norm. The inner product in $L^2(\Omega)$ is

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx.$$

Let $T > 0$ and $J = [0, T]$, $Q_T = J \times \Omega$, and $\Sigma_T = J \times \partial\Omega$. For a normed space E, the Bochner space $L^P(J; E)$ is defined with its standard norm.

We consider a second-order elliptic operator $\mathcal{L}$ in divergence form:

$$\mathcal{L}u = -\sum_{i,k=1}^{d} a_{ik}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} + \sum_{k=1}^{d} b_k(x)\frac{\partial u}{\partial x_k} + a_0(x)u, \tag{2.1}$$

where the coefficients are sufficiently smooth. We assume uniform ellipticity: there exists $\alpha_0 > 0$ such that for almost every $x \in \Omega$ and for every vector $\xi \in \mathbb{R}^d$, we have

$$\sum_{i,k=1}^{d} a_{ik}\xi_i\xi_k \geq \alpha_0 |\xi|^2, \tag{2.2}$$

and that

$$a_0(x) \geq \beta > 0, \quad \text{for a.e. } x \in \Omega. \tag{2.3}$$

The associated bilinear form $a(\cdot,\cdot)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ is defined by

$$a(u,v) = \int_\Omega \left(\sum_{i,k=1}^{d} a_{ik}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{d} b_k(x)\frac{\partial u}{\partial x_k}v + a_0(x)uv \right) dx. \tag{2.4}$$

This bilinear form is continuous and, under the given assumptions, can be made coercive (see Proposition 2.1 below).

Proposition 2.1. *If the bilinear form $a(\cdot,\cdot)$ associated with the uniformly elliptic operator $\mathcal{A}$ is continuous but not coercive on $H_0^1(\Omega)$, then there exists a constant $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the modified bilinear form*

$$b(\cdot,\cdot) = a(\cdot,\cdot) + \lambda(\cdot,\cdot) \tag{2.5}$$

is coercive on $H_0^1(\Omega)$, i.e., there exists a constant $\gamma > 0$ such that

$$b(v,v) \geq \gamma \|v\|_{H^1(\Omega)}^2, \quad \textit{for a.e. } v \in H_0^1(\Omega). \tag{2.6}$$

Proof. Let $v \in H_0^1(\Omega)$. By the definition of $a(\cdot,\cdot)$, we have

$$a(u,v) = \int_\Omega \left(\sum_{i,k=1}^{d} a_{ik}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{d} b_k(x)\frac{\partial u}{\partial x_k}v + a_0(x)uv \right) dx.$$

We decompose this expression into three integrals:

1. $\mathbf{I}_1 = \int_\Omega \sum_{i,k=1}^{d} a_{ik}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_k}dx,$

2. $\mathbf{I}_2 = \int_\Omega \sum_{k=1}^{d} b_k(x)\frac{\partial v}{\partial x_k}v dx,$

3. $\mathbf{I}_3 = \int_\Omega a_0(x)v^2 dx.$

Thus,

$$a(v,v) = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3. \tag{2.7}$$

Now we need to calculate the integrals $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

1. **For $\mathbf{I}_1$:** By the uniform ellipticity assumption (2.3), we have

$$\sum_{i,k=1}^{d} a_{ik}\xi_i\xi_k \geq \alpha_0 \left|\xi\right|^2, \quad \text{a.e. in } \Omega, \text{ for all } \xi \in \mathbb{R}^d.$$

Applying this inequality with $\xi = \nabla v\left(x\right)$, we obtain

$$\sum_{i,k=1}^{d} a_{ik}\xi_i\xi_k \geq \alpha_0 \left|\nabla v\left(x\right)\right|^2.$$

Integrating over Ω yields

$$\mathbf{I}_1 \geq \alpha_0 \int_\Omega \left|\nabla v\left(x\right)\right|^2 dx = \alpha_0 \left\|\nabla v\right\|^2_{L^2(\Omega)}. \tag{2.8}$$

2. **For $\mathbf{I}_2$:** Define

$$\left\|b\right\|_\infty = \sum_{k=1}^{d} \left\|b_k\right\|_\infty,$$

where $\left\|b_k\right\|_\infty = \max_{x\in\Omega}\left|b_k(x)\right|$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
\left|\mathbf{I}_2\right| &= \left|\int_\Omega \sum_{k=1}^{d} b_k\left(x\right)\frac{\partial v}{\partial x_k} v dx\right| \\
&\leq \sum_{k=1}^{d} \int_\Omega \left|b_k\left(x\right)\right| \left|\frac{\partial v}{\partial x_k}\right| \left|v\right| dx \\
&\leq \sum_{k=1}^{d} \int_\Omega \max_{x\in\Omega}\left|b_k(x)\right| \left|\frac{\partial v}{\partial x_k}\right| \left|v\right| dx \\
&= \sum_{k=1}^{d} \left\|b_k\right\|_\infty \int_\Omega \left|\frac{\partial v}{\partial x_k}\right| \left|v\right| dx \\
&\leq \sum_{k=1}^{d} \left\|b_k\right\|_\infty \left(\int_\Omega \left|\frac{\partial v}{\partial x_k}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|v\right|^2 dx\right)^{\frac{1}{2}} \\
&= \left\|b\right\|_\infty \left(\int_\Omega \left|\frac{\partial v}{\partial x_k}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|v\right|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we obtain

$$\left|\mathbf{I}_2\right| \leq \left\|b\right\|_\infty \left\|\nabla v\right\|_{L^2(\Omega)} \left\|v\right\|_{L^2(\Omega)}.$$

Therefore,

$$\mathbf{I}_2 \geq -\left\|b\right\|_\infty \left\|\nabla v\right\|_{L^2(\Omega)} \left\|v\right\|_{L^2(\Omega)}. \tag{2.9}$$

3. **For $\mathbf{I}_3$:** From the assumption $a_0\left(x\right) \geq \beta > 0$, we directly get

$$\begin{aligned}
\mathbf{I}_3 &= \int_\Omega a_0\left(x\right) v^2 dx \\
&\geq \beta \int_\Omega v^2 dx.
\end{aligned}$$

Thus,

$$\mathbf{I}_3 \geq \beta \left\|v\right\|^2_{L^2(\Omega)}. \tag{2.10}$$

Combining the inequalities (2.8), (2.9), and (2.10), we obtain

$$a(v,v) \geq \alpha_0 \|\nabla v\|^2_{L^2(\Omega)} - \|b\|_\infty \|\nabla v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \beta \|v\|^2_{L^2(\Omega)}. \tag{2.11}$$

For any $\epsilon > 0$, Young's inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ gives

$$\|\nabla v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \frac{\epsilon}{2} \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon} \|v\|^2_{L^2(\Omega)}.$$

Applied to the cross-term in (2.11), this implies

$$-\|b\|_\infty \|\nabla v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \geq -\|b\|_\infty \left(\frac{\epsilon}{2} \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon} \|v\|^2_{L^2(\Omega)}\right).$$

Substituting into (2.11),

$$a(v,v) \geq \left(\alpha_0 - \frac{\epsilon}{2}\|b\|_\infty\right) \|\nabla v\|^2_{L^2(\Omega)} + \left(\beta - \frac{1}{2\epsilon}\|b\|_\infty\right) \|v\|^2_{L^2(\Omega)}. \tag{2.12}$$

By definition,

$$\begin{aligned} b(v,v) &= a(v,v) + \lambda(v,v) \\ &= a(v,v) + \lambda \int_\Omega v^2 dx \\ &= a(v,v) + \lambda \|v\|^2_{L^2(\Omega)}. \end{aligned}$$

From (2.12), we have

$$b(v,v) \geq \left(\alpha_0 - \frac{\epsilon\|b\|_\infty}{2}\right) \|\nabla v\|^2_{L^2(\Omega)} + \left(\lambda + \beta - \frac{\|b\|_\infty}{2\epsilon}\right) \|v\|^2_{L^2(\Omega)}. \tag{2.13}$$

Choose $\epsilon > 0$ such that the coefficient of $\|\nabla v\|^2_{L^2(\Omega)}$ is positive. For instance, take

$$\epsilon = \frac{\alpha_0}{\|b\|_\infty}.$$

Then

$$\alpha_0 - \frac{\epsilon\|b\|_\infty}{2} = \alpha_0 - \frac{\alpha_0}{2} = \frac{\alpha_0}{2} > 0.$$

With this choice, inequality (2.13) becomes

$$b(v,v) \geq \frac{\alpha_0}{2} \|\nabla v\|^2_{L^2(\Omega)} + \left(\lambda + \beta - \frac{\|b\|_\infty}{2\epsilon}\right) \|v\|^2_{L^2(\Omega)}. \tag{2.14}$$

Let

$$\lambda_0 = \frac{\|b\|^2_\infty}{2\alpha_0} - \beta.$$

Then, for all $\lambda > \lambda_0$, the coefficient of $\|v\|^2_{L^2(\Omega)}$ in (2.14) is positive.

Since $v \in H^1_0(\Omega)$, Poincaré's inequality guarantees the existence of a constant $C_P > 0$ such that

$$\|v\|^2_{L^2(\Omega)} \leq C_P \|\nabla v\|^2_{L^2(\Omega)}.$$

The norm $\|v\|_{H^1_0(\Omega)}$ is equivalent to $\|\nabla v\|_{L^2(\Omega)}$.

Therefore, there exists a constant $\gamma > 0$ (depending on α_0, $\|b\|_\infty$, β, λ) such that

$$\begin{aligned} b(v,v) &\geq \gamma \|\nabla v\|^2_{L^2(\Omega)} \\ &= \gamma \|v\|^2_{H_0^1(\Omega)}. \end{aligned}$$

This establishes the coercivity of $b(\cdot,\cdot)$ on $H_0^1(\Omega)$ for

$$\lambda > \lambda_0 > \frac{\|b\|^2_\infty}{2\alpha_0} - \beta.$$

Now, if

$$\beta - \frac{\|b\|^2_\infty}{2\alpha_0} \geq 0,$$

then $\lambda_0 \leq 0$ and $a(\cdot,\cdot)$ might already be coercive without adding λ. The proposition addresses the case where it is not, which requires $\lambda > \lambda_0 \geq 0$. □

2.2 Problem statement and variational formulation

We consider the following parabolic variational inequality:

Problem 1. Find $u : Q_T \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u - f(u) \leq 0, \quad \text{in } Q_T, \\ u \leq \psi, \quad \text{in } Q_T, \\ \left(\dfrac{\partial u}{\partial t} + \mathcal{L}u - f(u)\right)(u - \psi) = 0, \quad \text{in } Q_T, \\ u\,|_{t=0} = u_0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \Sigma_T, \end{cases}$$

where $u_0 \in L^2(\Omega)$, the obstacle $\psi \in L^2\left(0,T;W^{2,\infty}(\Omega)\right)$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear source term assumed to be non-decreasing and Lipschitz continuous:

$$|f(x) - f(y)| \leq L, \quad \forall x,y \in \mathbb{R}, \tag{2.15}$$

with the Lipschitz constant L satisfying

$$L < \beta. \tag{2.16}$$

The corresponding weak formulation is to find $u(t) \in \mathcal{K} = \left\{v \in H_0^1(\Omega) : v \leq \psi \text{ in } \Omega\right\}$ for a.e. $t \in (0,T]$ such that

$$\left(\frac{\partial u}{\partial t}(t), v - u(t)\right) + a(u(t), v - u(t)) \geq (f(u(t)), v - u(t)), \quad v \in \mathcal{K}, \tag{2.17}$$

with $u(0) = u_0$.

3 Numerical approximation

3.1 Spatial discretization

Let $\Im_h$ be a quasi-uniform triangulation of Ω with mesh size h. We define the finite element space $\mathcal{V}_h$ of continuous piecewise linear functions:

$$\mathcal{V}_h = \left\{ v_h \in C\left(\bar{\Omega}\right) : v_h|_{T_j} \in \mathcal{P}_1\left(T\right), \ \forall T \in \Im_h \ , \ v_h = 0 \text{ on } \partial\Omega \right\}. \tag{3.1}$$

Let $r_h : H^1\left(\Omega\right) \longrightarrow \mathcal{V}_h$ be the Lagrange interpolation operator. The discrete convex set is

$$\mathcal{K}_h = \left\{ v_h \in \mathcal{V}_h : v_h \leq r_h \psi \right\}.$$

The semi-discrete problem is: find $u_h\left(t\right) \in \mathcal{K}_h$ such that for a.e. $t \in [0, T]$,

$$\left(\frac{\partial u_h}{\partial t}\left(t\right), v_h - u_h\left(t\right) \right) + a\left(u_h\left(t\right), v_h - u_h\left(t\right)\right) \geq \left(f\left(u_h\left(t\right)\right), v_h - u_h\left(t\right)\right), \quad v_h \in \mathcal{K}_h, \tag{3.2}$$

with $u_h\left(0\right) = u_{0,h}$.

3.2 Fully discrete $\theta-$scheme

Let $\Delta t = \dfrac{T}{N}$ and $t_k = k\Delta t$ for $k = 0, 1, \ldots, N$. For a parameter $\theta \in [0, 1]$, we define

$$u_h^{k+\theta} = \theta u_h^{k+1} + \left(1 - \theta\right) u_h^k. \tag{3.3}$$

The fully discrete scheme is: for $k = 0, 1, \ldots, N - 1$, find $u_h^{k+1} \in \mathcal{K}_h$ such that

$$\left(\frac{u_h^{k+1} - u_h^k}{\Delta t}, v_h - u_h^{k+\theta} \right) + a\left(u_h^{k+\theta}, v_h - u_h^{k+\theta}\right) \geq \left(f\left(u_h^{k+\theta}\right), v_h - u_h^{k+\theta}\right), \quad v_h \in \mathcal{K}_h. \tag{3.4}$$

4 L^2-stability analysis

The following theorem is the main result of our stability analysis.

Theorem 4.1. *Assume that the bilinear form* $a\left(\cdot, \cdot\right)$ *is continuous and coercive with constants* M *and* γ*, respectively, and that* f *is Lipschitz with constant* L*. Then, the* $\theta-$*scheme (3.4) satisfies the following stability properties:*

1. *For* $\theta \in \left[\frac{1}{2}, 1\right]$*, the scheme is unconditionally stable in the* L^2-*norm.*

2. *For* $\theta \in \left[0, \frac{1}{2}\right)$*, the scheme is stable under the condition*

$$\Delta t \leq \frac{2}{L\left(1 - \theta\left(1 - \theta\right)\right)}$$

Proof. Let u_h^k and u_h^{k+1} be two consecutive solutions of the scheme (3.4). Since both reside in $\mathcal{K}_h$, $v_h = u_h^k$ is an admissible test function in (3.4) for the time step $k + 1$.

Choosing this test function yields

$$\left(\frac{u_h^{k+1}-u_h^k}{\Delta t},\ u_h^k-u_h^{k+\theta}\right)+a\left(u_h^{k+\theta},\ u_h^k-u_h^{k+\theta}\right)\geq\left(f\left(u_h^{k+\theta}\right),\ u_h^k-u_h^{k+\theta}\right). \tag{4.1}$$

Let us denote

$$\mathbf{e}^{k+1}=u_h^{k+1}-u_h^k.$$

As $u_h^{k+\theta}=\theta u_h^{k+1}+(1-\theta)\,u_h^k$, then we have:

$$u_h^k-u_h^{k+\theta}=-\theta\ \mathbf{e}^{k+1}. \tag{4.2}$$

So, the first term in (4.1) becomes:

$$\begin{aligned}\left(\frac{u_h^{k+1}-u_h^k}{\Delta t},\ u_h^k-u_h^{k+\theta}\right) &= \frac{1}{\Delta t}\left(u_h^{k+1}-u_h^k,\ u_h^k-u_h^{k+\theta}\right)\\ &= \frac{1}{\Delta t}\left(\mathbf{e}^{k+1},\ -\theta\ \mathbf{e}^{k+1}\right)\\ &= -\frac{\theta}{\Delta t}\left(\mathbf{e}^{k+1},\ \mathbf{e}^{k+1}\right)\\ &= -\frac{\theta}{\Delta t}\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2.\end{aligned}$$

Substituting into (4.1) and multiplying by Δt, we obtain:

$$-\theta\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2+\Delta t\ a\left(u_h^{k+\theta},\ u_h^k-u_h^{k+\theta}\right)\geq\Delta t\ \left(f\left(u_h^{k+\theta}\right),\ u_h^k-u_h^{k+\theta}\right). \tag{4.3}$$

Notice that

$$a\left(u_h^{k+\theta},\ u_h^k-u_h^{k+\theta}\right)=-a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)+a\left(u_h^{k+\theta},\ u_h^k\right)$$

and

$$\left(f\left(u_h^{k+\theta}\right),\ u_h^k-u_h^{k+\theta}\right)=-\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)+\left(f\left(u_h^{k+\theta}\right),\ u_h^k\right).$$

Hence

$$\Delta t\ \left(-a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)+a\left(u_h^{k+\theta},\ u_h^k\right)\right)+\Delta t\ \left(\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)-\left(f\left(u_h^{k+\theta}\right),\ u_h^k\right)\right)\geq\theta\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2. \tag{4.4}$$

The left-hand side can be interpreted as an energy difference. We now proceed case by case.

Case 1: $\theta\in\left[\frac{1}{2},\ 1\right]$

Using the coercivity of $a\,(.,\ .)$, standard estimate yield:

$$a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)\geq\gamma\left\|u_h^{k+\theta}\right\|_{H_0^1(\Omega)}^2,$$

then

$$-a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)\leq-\gamma\left\|u_h^{k+\theta}\right\|_{H_0^1(\Omega)}^2.$$

Now, we use the continuity of the bilinear form $a(.,\ .)$ and Young's inequality to estimate $a\left(u_h^{k+\theta},\ u_h^k\right)$:

$$\begin{aligned}\left|a\left(u_h^{k+\theta},\ u_h^k\right)\right| &\leq M\ \left\|u_h^{k+\theta}\right\|_{H_0^1(\Omega)}\ \left\|u_h^k\right\|_{H_0^1(\Omega)}\\ &\leq \frac{M}{2}\ \left(\left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}+\left\|u_h^k\right\|^2_{H_0^1(\Omega)}\right).\end{aligned}$$

Thus

$$-a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)+a\left(u_h^{k+\theta},\ u_h^k\right)\leq\left(\frac{M}{2}-\gamma\right)\left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}+\frac{M}{2}\ \left\|u_h^k\right\|^2_{H_0^1(\Omega)}. \tag{4.5}$$

Now, we apply the Lipschitz boundedness of f, Cauchy- Schwarz inequality and Young's inequality with parameter $\epsilon=1$ to estimate $\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)$-$\left(f\left(u_h^{k+\theta}\right),\ u_h^k\right)$:

Note that

$$\begin{aligned}\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)-\left(f\left(u_h^{k+\theta}\right),\ u_h^k\right) &= \left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}-u_h^k\right)\\ &= -\theta\left(f\left(u_h^{k+\theta}\right),\ e^{k+1}\right).\end{aligned}$$

Hence

$$\left|\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)-\left(f\left(u_h^{k+\theta}\right),\ u_h^k\right)\right|\leq\theta\frac{L}{2}\left(\left\|u_h^{k+\theta}\right\|^2_{L^2(\Omega)}+\left\|e^{k+1}\right\|^2_{L^2(\Omega)}\right). \tag{4.6}$$

Combining $(4.4)-(4.6)$, we obtain:

$$\theta\left(1-\Delta t\frac{L}{2}\right)\ \left\|e^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \left(\gamma-\frac{M}{2}\right)\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\leq\Delta t\ \frac{M}{2}\ \left\|u_h^k\right\|^2_{H_0^1(\Omega)}+\Delta t\ \frac{\theta L}{2}\left\|u_h^{k+\theta}\right\|^2_{L^2(\Omega)}. \tag{4.7}$$

Using the identity:

$$\left\|u_h^{k+\theta}\right\|^2_{L^2(\Omega)}=\theta\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}+(1-\theta)\left\|u_h^k\right\|^2_{L^2(\Omega)}-\theta(1-\theta)\left\|\mathbf{e}^{k+1}\right\|^2_{L^2(\Omega)}. \tag{4.8}$$

Thus

$$\Delta t\ \frac{\theta L}{2}\left\|u_h^{k+\theta}\right\|^2_{L^2(\Omega)}=\Delta t\ \frac{\theta^2 L}{2}\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \frac{\theta(1-\theta)L}{2}\left\|u_h^k\right\|^2_{L^2(\Omega)}-\Delta t\ \frac{\theta^2(1-\theta)L}{2}\left\|\mathbf{e}^{k+1}\right\|^2_{L^2(\Omega)}. \tag{4.9}$$

Combining (4.7) with (4.9), we obtain:

$$\begin{aligned}&\theta\left(1-\Delta t\frac{L}{2}\left(1-\ \theta(1-\theta)\right)\right)\ \left\|e^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \left(\gamma-\frac{M}{2}\right)\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\\ \leq&\ \Delta t\ \frac{M}{2}\ \left\|u_h^k\right\|^2_{H_0^1(\Omega)}+\Delta t\ \frac{\theta L}{2}\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \frac{\theta(1-\theta)L}{2}\left\|u_h^k\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Observe that, for $\theta\geq\frac{1}{2}$:

$$1-\ \theta(1-\theta)\geq\frac{3}{4}.$$

So, the coefficient of $\|e^{k+1}\|^2_{L^2(\Omega)}$ remains strictly positive for all Δt.

Let us denote:

$$\theta\left(1-\Delta t\frac{L}{2}\left(1+\ \theta\left(1-\theta\right)\right)\right)=C_0.$$

We obtain an inequality of the form:

$$\begin{aligned}&C_0\left\|e^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \left(\gamma-\frac{M}{2}\right)\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\\ \leq\ &\Delta t\ \frac{M}{2}\ \left\|u_h^k\right\|^2_{H_0^1(\Omega)}+\Delta t\ \frac{\theta L}{2}\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \frac{\theta\left(1-\theta\right)L}{2}\left\|u_h^k\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the following inequality:

$$\left\|e^{k+1}\right\|^2_{L^2(\Omega)}\geq\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}-\left\|u_h^k\right\|^2_{L^2(\Omega)},$$

we obtain:

$$\begin{aligned}&C_0\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}-C_0\left\|u_h^k\right\|^2_{L^2(\Omega)}+\Delta t\ \left(\gamma-\frac{M}{2}\right)\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\\ \leq\ &\Delta t\ \frac{M}{2}\ \left\|u_h^k\right\|^2_{H_0^1(\Omega)}+\Delta t\ \frac{\theta L}{2}\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ \frac{\theta\left(1-\theta\right)L}{2}\left\|u_h^k\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Using Poincaré inequality gives:

$$\begin{aligned}&C_0\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}-C_0\left\|u_h^k\right\|^2_{L^2(\Omega)}+C_1\ \Delta t\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\\ \leq\ &\Delta t\ C_2\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}+\Delta t\ C_3\left\|u_h^k\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus

$$\left(C_0-\Delta t\ C_2\right)\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}-C_0\left\|u_h^k\right\|^2_{L^2(\Omega)}+\Delta t\ C_1\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\leq\Delta t\ C_3\left\|u_h^k\right\|^2_{L^2(\Omega)}$$

where $C_1,\ C_2,\ C_3>0$ independent of of Δt.and h.

As $\theta>\frac{1}{2},$ then the coefficient of $\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}$:

$$\left(C_0-\Delta t\ C_2\right)>0,$$

then, this term is absorbed into the left-hand side, yielding:

$$\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}-\left\|u_h^k\right\|^2_{L^2(\Omega)}+\Delta t\ C_1\ \left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\leq\Delta t\ C_3\left\|u_h^k\right\|^2_{L^2(\Omega)}.$$

Summing for $k=0,...,N-1$ gives

$$\sum_{k=0}^{N-1}\left(\left\|u_h^{k+1}\right\|^2_{L^2(\Omega)}-\left\|u_h^k\right\|^2_{L^2(\Omega)}\right)+\Delta t\ C_1\ \sum_{k=0}^{N-1}\left\|u_h^{k+\theta}\right\|^2_{H_0^1(\Omega)}\leq\Delta t\ C_3\ \sum_{k=0}^{N-1}\left\|u_h^k\right\|^2_{L^2(\Omega)}.$$

Using the indentity:

$$\sum_{k=0}^{N-1}\left(\left\|u_h^{k+1}\right\|_{L^2(\Omega)}^2-\left\|u_h^{k}\right\|_{L^2(\Omega)}^2\right)=\|u_h^n\|_{L^2(\Omega)}^2-\|u_h^0\|_{L^2(\Omega)}^2,$$

gives:

$$\|u_h^n\|_{L^2(\Omega)}^2-\|u_h^0\|_{L^2(\Omega)}^2+\Delta t\ C_1\sum_{k=0}^{N-1}\left\|u_h^{k+\theta}\right\|_{H_0^1(\Omega)}^2\leq\Delta t\ C_3\sum_{k=0}^{N-1}\left\|u_h^{k}\right\|_{L^2(\Omega)}^2.$$

Applying the discrete Grönwall lemma, we obtain:

$$\|u_h^n\|_{L^2(\Omega)}^2\leq C\left\|u_h^0\right\|_{L^2(\Omega)}^2.$$

where $C>0$ is a constant independent of Δt.and h.

Consequently, we have:

$$\sup_{0\leq k\leq N}\|u_h^n\|_{L^2(\Omega)}\leq C\left\|u_h^0\right\|_{L^2(\Omega)}.$$

This proves unconditional stability.

Case 2: $\theta\in\left[0,\ \frac{1}{2}\right)$

The term $\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2$ must be positive and controlled. We return to the fundamental inequality (4.4) :

$$\Delta t\ \left(-a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)+a\left(u_h^{k+\theta},\ u_h^{k}\right)+\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)-\left(f\left(u_h^{k+\theta}\right),\ u_h^{k}\right)\right)\geq\theta\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2.$$

Or equivalently:

$$-a\left(u_h^{k+\theta},\ u_h^{k+\theta}\right)+a\left(u_h^{k+\theta},\ u_h^{k}\right)+\left(f\left(u_h^{k+\theta}\right),\ u_h^{k+\theta}\right)-\left(f\left(u_h^{k+\theta}\right),\ u_h^{k}\right)\geq\frac{\theta}{\Delta t}\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2.\tag{4.10}$$

Using coercivity, continuity, Lipschitz continuity of f, and Young's inequality, we can show that:

$$\theta\left(1-\Delta t\ \frac{L}{2}\right)\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2\leq\ \Delta t\ C_4\ \left\|u_h^{k+\theta}\right\|_{H_0^1(\Omega)}^2+\Delta t\ C_5\ \left\|u_h^{k}\right\|_{H_0^1(\Omega)}^2+\Delta t\ \frac{\theta L}{2}\ \left\|u_h^{k+\theta}\right\|_{L^2(\Omega)}^2\tag{4.11}$$

where $C_4,\ C_5>0$ independent of Δt and h.

Using the identity:

$$\left\|u_h^{k+\theta}\right\|_{L^2(\Omega)}^2=\theta\left\|u_h^{k+1}\right\|_{L^2(\Omega)}^2+(1-\theta)\left\|u_h^{k}\right\|_{L^2(\Omega)}^2-\theta\,(1-\theta)\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2.\tag{4.12}$$

By substituting (4.12) into (4.11), , the coefficient of $\left\|\mathbf{e}^{k+1}\right\|_{L^2(\Omega)}^2$ becomes:

$$\theta\left(1-\Delta t\ \frac{L}{2}+\Delta t\ \frac{L}{2}\theta\,(1-\theta)\right).$$

Observe that for $\theta<\frac{1}{2}$:

$$\theta\,(1-\theta)>\theta^2.$$

So, the coefficient of $\left\|\mathbf{e}^{k+1}\right\|^2_{L^2(\Omega)}$ can become negative if Δt is too large, i.e.,

$$1 - \Delta t\,\frac{L}{2} + \Delta t\,\frac{L}{2}\theta\,(1-\theta) < 0.$$

Then, in order to guarantee the stability of the scheme, it is necessary to impose:

$$\Delta t < \frac{2}{L\,(1-\theta\,(1-\theta))},$$

which completes the proof. □

5 Conclusion

This work presents an L^2-stability analysis of a $\theta-$scheme applied to a nonlinear parabolic variational inequality of obstacle type. It has been established that the scheme is unconditionally stable when the implicitness parameter satisfies $\theta \geq \frac{1}{2}$. Conversely, for $\theta < \frac{1}{2}$, stability requires a Courant–Friedrichs–Lewy (CFL)-type condition linking the time step, the coercivity constant, and the Lipschitz constant of the nonlinearity. These results provide a rigorous framework for the optimal choice of numerical parameters in the simulation of obstacle problems. Future directions include extending the analysis to higher-order schemes and other types of constraints.

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Authors' contributions

M. A. Bencheikh Le Hocine: conceptualization, methodology, formal analysis, writing–original draft. Y. Bellour: validation, writing–review & editing.

Conflict of interest

The authors have no conflicts of interest to declare.

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