

Approximate solution of linear Volterra-Fredholm integral equations via exponential spline function

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Abstract. This paper presents a novel numerical scheme for solving linear Volterra-Fredholm integral equations (V-FIEs) of the second kind, utilizing exponential spline functions (ESFs) in combination with fractional derivatives. The method simplifies computational implementation by converting the original integral equation into a matrix system. To prove the precision and stability of the suggested approach, a thorough convergence analysis is carried out. Numerical experiments, backed by graphical representations, validate the method's high accuracy and computational efficiency, even with a limited number of subintervals. All simulations and visualizations are implemented using Python. The results indicate that the suggested ESF approach performs noticeably better than traditional methods.

Keywords: Exponential spline functions, Linear integral equations, Approximate solutions, Natural spline initial conditions.

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1 Introduction

Integral equations are fundamental in expressing a wide variety of problems in mathematical physics. Although an exhaustive compilation of their applications is impractical, this paper will explicitly examine and treat specific examples. It is no exaggeration to state that integral equations are integral to nearly every field of applied mathematics and mathematical physics; thus, the body of literature on integral equations and their applications is vast.

Recent research has extensively investigated the interplay among Fredholm integral equations, Volterra integral equations, mixed Volterra-Fredholm integral equations, and their respective numerical treatments.

This work focuses on linear mixed Volterra-Fredholm integral equations of the form:

$$\varphi(z) = f(z) + \lambda_1 \int_{a_0}^{a_1} \psi(z, t)\varphi(t) dt + \lambda_2 \int_{a_0}^z \psi(z, t)\varphi(t) dt, \quad \text{for } z \in I, \quad (1.1)$$

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where the unknown function φ is to be determined. The functions $f \in C[I, \mathbb{R}]$ and $\psi \in C[S, \mathbb{R}]$ are known, with $S = \{(z, t) : a_0 \leq z \leq t \leq a_1\}$. The parameters λ_1 and λ_2 are constants, $I = [a_0, a_1]$, and we assume $\varphi \in C^2[I, \mathbb{R}]$.

Several investigations have previously utilized various techniques with suitable accuracy and efficiency for analyzing Volterra-Fredholm integral equations. For instance, [2] used hyperbolic basis functions to solve second-kind linear Volterra-Fredholm integral equations. In [6], the authors employed non-polynomial spline basis and quasi-linearization methods to solve nonlinear Volterra integral equations. [8] used shifted Legendre polynomials approximation for solving Volterra-Fredholm integral equations, while [3] employed an expansion method based on the composition of inverse and direct discrete fuzzy transforms for Volterra-Fredholm integral equations. Fractional calculus is used to solve problems in mathematics, physical science, engineering, and computer science. There are many definitions of fractional operators, such as Caputo’s fractional integrals and derivatives, Grünwald-Letnikov’s, Riemann-Liouville’s [1,4,5,7,9–12].

The paper is organized as follows: Section 2 introduces the exponential spline function. In Section 3, this function is used to solve the Volterra-Fredholm integral equation and obtain a matrix form. In Section 4, lemmas and theorems are proven through numerical examples to establish the convergence of the proposed method. In Section 5, the applicability and validity of the new method are illustrated. Finally, concluding remarks are given in Section 6.

2 Exponential spline function

Consider a uniform mesh Δ with nodal points on $[a_0, a_1]$ such that

$$\Delta : a_0 = t_0 < t_1 < \dots < t_n = a_1,$$

where $h = \frac{a_1 - a_0}{n}$. The interpolating exponential spline function $S_j(t)$ interpolates φ at t_j :

$$S_j(t) = a_j e^{\lambda\tau(t-t_j)} + b_j e^{\frac{\lambda\tau(t-t_j)}{2}} + c_j e^{\frac{\lambda\tau(t-t_j)}{4}} + d_j e^{\frac{\lambda\tau(t-t_j)}{8}}, \quad j = 0, 1, \dots, n, \tag{2.1}$$

where λ and τ are arbitrary constants. The following boundary conditions are used to derive the approximate solution of equation (1.1):

$$S_j(t_j) = \varphi_j, \quad S_j(t_{j+1}) = \varphi_{j+1}, \quad S'_j(t_j) = M_j, \quad S'_j(t_{j+1}) = M_{j+1}. \tag{2.2}$$

Through algebraic manipulation of equations (2.1) and (2.2), we obtain

$$\begin{aligned}
a_j &= \frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)\lambda\tau\varphi_j - (2\alpha_8 - 3\alpha_9 + \alpha_{10})\lambda\tau\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \\
&\quad - \frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau}, \\
b_j &= \frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\lambda\tau\varphi_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\lambda\tau\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \\
&\quad + \frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau}, \\
c_j &= \frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\lambda\tau\varphi_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\lambda\tau\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \\
&\quad + \frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau}, \\
d_j &= \frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\lambda\tau\varphi_j + (8\alpha_{11} - 12\alpha_8 + 4)\lambda\tau\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \\
&\quad + \frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau}.
\end{aligned}$$

where $\alpha_0 = e^{\frac{3\lambda\tau h}{4}}$, $\alpha_1 = e^{\frac{5\lambda\tau h}{4}}$, $\alpha_3 = e^{\frac{3\lambda\tau h}{8}}$, $\alpha_4 = e^{\frac{9\lambda\tau h}{8}}$, $\alpha_6 = e^{\frac{5\lambda\tau h}{8}}$, $\alpha_7 = e^{\frac{3\lambda\tau h}{2}}$, $\alpha_8 = e^{\frac{\lambda\tau h}{2}}$, $\alpha_9 = e^{\frac{\lambda\tau h}{4}}$, $\alpha_{10} = e^{\frac{\lambda\tau h}{8}}$, and $\alpha_{11} = e^{\lambda\tau h}$.

From $S_j^{(\frac{1}{2})}(t_j) = S_{j-1}^{(\frac{1}{2})}(t_j)$, we obtain the following system:

$$\delta_0 M_{j-1} + \delta_1 M_j + \delta_2 M_{j+1} = \tau[\gamma_0 \varphi_{j-1} + \gamma_1 \varphi_j + \gamma_2 \varphi_{j+1}], \quad (2.3)$$

where

$$\begin{aligned}
\alpha_2 &= e^{\frac{7\lambda\tau h}{8}}, \quad \alpha_5 = e^{\frac{11\lambda\tau h}{8}}, \quad \beta_0 = e^{\frac{-3\lambda\tau h}{8}}, \quad \beta_1 = e^{\frac{-\lambda\tau h}{4}}, \quad \beta_2 = e^{\frac{-5\lambda\tau h}{8}}, \quad \beta_3 = e^{\frac{-\lambda\tau h}{8}}, \\
\beta_4 &= e^{\frac{-3\lambda\tau h}{4}}, \quad \beta_5 = e^{\frac{-7\lambda\tau h}{8}}, \quad \beta_6 = e^{\frac{-\lambda\tau h}{2}}, \\
\delta_0 &= \left(12\beta_0 - 8\beta_1 - 4\beta_2 + 12\sqrt{2}\alpha_0 + \sqrt{2^3}\beta_3 - \sqrt{4^3}\alpha_1 + 7\sqrt{4}\alpha_2 - 3\sqrt{4}\alpha_3 \right. \\
&\quad \left. - 3\sqrt{8}\alpha_4 + 2\sqrt{8}\alpha_5 - 3\sqrt{2^5}\alpha_6\right), \\
\delta_1 &= \left(\left(2 - \sqrt{2^3}\right)\alpha_0 - 6\alpha_6 + \left(4 - \sqrt{2^7}\right)\alpha_3 + \left(-14 + 7\sqrt{2^3}\right)\alpha_4 - 3\sqrt{2^3}\alpha_1 \right. \\
&\quad \left. + \left(8 - \sqrt{2^5}\right)\alpha_7 + 12\beta_4 - 8\beta_5 - 4\beta_6 - 14\sqrt{2}\beta_1 + \sqrt{2^3}\alpha_8 + 12\sqrt{2}\beta_0 \right. \\
&\quad \left. - \sqrt{4^3}\beta_3 + 7\sqrt{4}\alpha_9 + 2\sqrt{8}\alpha_{10} + \sqrt{8}\alpha_2\right), \\
\delta_2 &= \left(-10\sqrt{2}\beta_0 - 9\alpha_1 - 2\beta_6 + \beta_4 - \left(5\sqrt{2^3} - 12\right)\alpha_9 + \left(3\sqrt{2^5} + 16\right)\alpha_{10} \right. \\
&\quad \left. + \left(3\sqrt{2^3} - 10\right)\alpha_8 - \left(\sqrt{2^5} + 10\right)\alpha_{11} - 3\sqrt{2^5}\alpha_6 + 10\sqrt{4}\alpha_2 + 5\alpha_3\right),
\end{aligned}$$

$$\begin{aligned} \gamma_0 &= \left(3\beta_0 - \frac{3}{\sqrt{2}}\alpha_6 - 2\beta_2 - \beta_1 + \sqrt{2^3}\beta_3 + \frac{3}{\sqrt{2}}\alpha_0 + \frac{14}{\sqrt{4}}\alpha_2 \right. \\ &\quad \left. - 3\sqrt{4}\alpha_3 - 2\alpha_1 + \frac{4}{\sqrt{8}}\alpha_5 - \frac{12}{\sqrt{8}}\alpha_{10} \right), \\ \gamma_1 &= \left((10 + \sqrt{2^3})\alpha_3 + (5 + \sqrt{2^3})\alpha_0 - 3\alpha_6 + \frac{3}{\sqrt{2}}\alpha_1 \right. \\ &\quad \left. + (7 - \frac{7}{\sqrt{2}})\alpha_4 + (\sqrt{2} - 1)\alpha_7 + (3\sqrt{2} + 7)\alpha_9 - 3\beta_3 - \sqrt{2^3}\alpha_8 \right. \\ &\quad \left. - \sqrt{8}\alpha_2 + 2\beta_6 - 3\beta_4 + \beta_5 + \frac{7}{\sqrt{2}}\beta_1 - \frac{3}{\sqrt{2}}\beta_0 \right), \\ \gamma_2 &= \left(7\alpha_8 + 4\alpha_9 + \left(\frac{\sqrt{2^3} - 3}{\sqrt{2}} \right) \alpha_{10} + (6 - \sqrt{2^5})\alpha_{11} - 4 \right). \end{aligned}$$

With the natural spline initial condition $M_0 = M_n = 0$, equation (2.3) has a unique solution. To obtain M_1, \dots, M_{n-1} , the system can be written in matrix form:

$$M = A^{-1}B\Phi = \hat{F}\Phi, \tag{2.4}$$

where $M = (M_0, M_1, \dots, M_n)^T$, A^{-1} is a diagonal matrix formed from δ_i , B is a diagonal matrix formed from γ_i for $i = 0, 1, 2$, and $\Phi = (\varphi_0, \varphi_1, \dots, \varphi_n)^T$.

3 Methodology

The exponential spline function proposed in Section 2 is used to approximate equation (1.1):

$$\begin{aligned} \varphi(z_i) &= f(z_i) + \lambda_1 \int_{a_0}^{a_1} \psi(z_i, t)\varphi(t) dt + \lambda_2 \int_{a_0}^{z_i} \psi(z_i, t)\varphi(t) dt \\ &= f(z_i) + \lambda_1 \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \psi(z_i, t)\varphi(t) dt + \lambda_2 \sum_{j=0}^{i-1} \int_{z_j}^{z_{j+1}} \psi(z_i, t)\varphi(t) dt \\ &\approx f(z_i) + \lambda_1 \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \psi(z_i, t)S_j(t) dt + \lambda_2 \sum_{j=0}^{i-1} \int_{z_j}^{z_{j+1}} \psi(z_i, t)S_j(t) dt + O(h^4). \end{aligned}$$

$$\begin{aligned}
&= f(z_i) + \lambda_1 \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \psi(z_i, t) [a_j e^{\lambda\tau(t-t_j)} + b_j e^{\frac{\lambda\tau(t-t_j)}{2}} + c_j e^{\frac{\lambda\tau(t-t_j)}{4}} + d_j e^{\frac{\lambda\tau(t-t_j)}{8}}] dt \\
&+ \lambda_2 \sum_{j=0}^{i-1} \int_{z_j}^{z_{j+1}} \psi(z_i, t) [a_j e^{\lambda\tau(t-t_j)} + b_j e^{\frac{\lambda\tau(t-t_j)}{2}} + c_j e^{\frac{\lambda\tau(t-t_j)}{4}} + d_j e^{\frac{\lambda\tau(t-t_j)}{8}}] dt + O(h^4) \\
&= f(z_i) + \lambda_1 \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \psi(z_i, t) \left[\left(\frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)\varphi_j - (2\alpha_8 - 3\alpha_9 + \alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right. \\
&\quad \left. \left. - \frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\lambda\tau(t-t_j)} \right. \\
&\quad \left. + \left(\frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\varphi_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right. \\
&\quad \left. \left. + \frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{2}} \right. \\
&\quad \left. + \left(\frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\varphi_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right. \\
&\quad \left. \left. + \frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{4}} \right. \\
&\quad \left. + \left(\frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\varphi_j + (8\alpha_{11} - 12\alpha_8 + 4)\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right. \\
&\quad \left. \left. + \frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{8}} \right] dt \\
&+ \lambda_2 \sum_{j=0}^{i-1} \int_{z_j}^{z_{j+1}} \psi(z_i, t) \left[\left(\frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)\varphi_j - (2\alpha_8 - 3\alpha_9 + \alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right. \\
&\quad \left. \left. - \frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\lambda\tau(t-t_j)} \right. \\
&\quad \left. + \left(\frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\varphi_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right. \\
&\quad \left. \left. + \frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{2}} \right. \\
&\quad \left. + \left(\frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\varphi_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & + \frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \Big) e^{\frac{\lambda\tau(t-t_j)}{4}} \\
 & + \left(\frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\varphi_j + (8\alpha_{11} - 12\alpha_8 + 4)\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right. \\
 & \left. + \frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{8}} \Big] dt + O(h^4).
 \end{aligned}$$

The following integral parts have been solved by trapezoidal integration rule:

$$\begin{aligned}
 \varphi(z_i) = f(z_i) & + \underbrace{\sum_{j=0}^{i-1} \left[\frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)\varphi_j - (2\alpha_8 - 3\alpha_9 + \alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\lambda\tau(t-t_j)} dt}_{a_{i,j}} \\
 & - \sum_{j=0}^{i-1} \left[\frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\lambda\tau(t-t_j)} dt \\
 & + \underbrace{\sum_{j=0}^{i-1} \left[\frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\varphi_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{2}} dt}_{b_{i,j}} \\
 & + \sum_{j=0}^{i-1} \left[\frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{2}} dt \\
 & + \sum_{j=0}^{i-1} \left[\frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\varphi_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \underbrace{\int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{4}} dt}_{c_{i,j}} \\
 & + \sum_{j=0}^{i-1} \left[\frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{4}} dt \\
 & + \underbrace{\sum_{j=0}^{i-1} \left[\frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\varphi_j + (8\alpha_{11} - 12\alpha_8 + 4)\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{8}} dt}_{d_{i,j}} \\
 & + \sum_{j=0}^{i-1} \left[\frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_1 \int_{t_j}^{t_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{8}} dt \\
 & + \underbrace{\sum_{j=0}^{i-1} \left[\frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)\varphi_j - (2\alpha_8 - 3\alpha_9 + \alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\lambda\tau(t-t_j)} dt}_{e_{i,j}} \\
 & - \sum_{j=0}^{i-1} \left[\frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\lambda\tau(t-t_j)} dt \\
 & + \underbrace{\sum_{j=0}^{i-1} \left[\frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\varphi_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{2}} dt}_{v_{i,j}} \\
 & + \sum_{j=0}^{i-1} \left[\frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{2}} dt
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{i-1} \left[\frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\varphi_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \underbrace{\lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{4}} dt}_{r_{i,j}} \\
& + \sum_{j=0}^{i-1} \left[\frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{4}} dt \\
& + \sum_{j=0}^{i-1} \left[\frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\varphi_j + (8\alpha_{11} - 12\alpha_8 + 4)\varphi_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \underbrace{\lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{8}} dt}_{s_{i,j}} \\
& + \sum_{j=0}^{i-1} \left[\frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right] \lambda_2 \int_{z_j}^{z_{j+1}} \psi(z_i, t) e^{\frac{\lambda\tau(t-t_j)}{8}} dt + O(h^4). \\
& = f(z_i) + \frac{1}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3} \left(\sum_{j=0}^{i-1} \left[(3\alpha_6 - 2\alpha_3 - \alpha_0)\varphi_j - (12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j \right] a_{i,j} \right. \\
& - \sum_{j=0}^{i-1} \left[(2\alpha_8 - 3\alpha_9 + \alpha_{10})\varphi_{j+1} + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1} \right] a_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(4\alpha_3 - 3\alpha_1 - 7\alpha_4)\varphi_j + (28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j \right] b_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\varphi_{j+1} + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1} \right] b_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\varphi_j + (16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j \right] c_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\varphi_{j+1} + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1} \right] c_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\varphi_j + (24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j \right] d_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(8\alpha_{11} - 12\alpha_8 + 4)\varphi_{j+1} + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1} \right] d_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(3\alpha_6 - 2\alpha_3 - \alpha_0)\varphi_j - (12\alpha_6 - 8\alpha_0 - 4\alpha_3)M_j \right] e_{i,j} \\
& - \sum_{j=0}^{i-1} \left[(2\alpha_8 - 3\alpha_9 + \alpha_{10})\varphi_{j+1} + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)M_{j+1} \right] e_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\varphi_j + (28\alpha_4 - 24\alpha_1 - 4\alpha_3)M_j \right] v_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\varphi_{j+1} + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})M_{j+1} \right] v_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\varphi_j + (16\alpha_7 - 28\alpha_4 + 12\alpha_6)M_j \right] r_{i,j}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{i-1} \left[(14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\varphi_{j+1} + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})M_{j+1} \right] r_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\varphi_j + (24\alpha_1 - 8\alpha_0 - 16\alpha_7)M_j \right] g_{i,j} \\
& + \sum_{j=0}^{i-1} \left[(8\alpha_{11} - 12\alpha_8 - 4)\varphi_{j+1} + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})M_{j+1} \right] g_{i,j} \Big) + O(h^4).
\end{aligned}$$

Suppose that $a_{i,n} = b_{i,0} = c_{i,n} = d_{i,0} = e_{i,n} = v_{i,0} = h_{i,n} = g_{i,0} = 0$, and $A = a_{i,j}, B = b_{i,j}, C = c_{i,j}, D = d_{i,j}, E = e_{i,j}, V = v_{i,j}, R = r_{i,j}, G = g_{i,j}, F = (f_0, f_1, \dots, f_N)^T, M = (M_0, M_1, \dots, M_N)^T, \Phi = (\varphi_0, \varphi_1, \dots, \varphi_N)^T$. Substituting the spline representation and simplifying leads to the matrix equation:

$$\Phi = F + \frac{\hat{F}_0}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3} \Phi + \frac{\hat{F}_1}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3} M, \quad (3.1)$$

where

$$\begin{aligned}
\hat{F}_0 = & \left[((3\alpha_6 - 2\alpha_3 - \alpha_0) - (2\alpha_8 - 3\alpha_9 + \alpha_{10}))A + ((4\alpha_3 - 3\alpha_1 - 7\alpha_4) + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10}))B + \right. \\
& ((14\alpha_4 - 12\alpha_6 - 2\alpha_7) + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11}))C + ((8\alpha_0 + 4\alpha_7 - 12\alpha_9) + (8\alpha_{11} - 12\alpha_8 + 4))D + \\
& ((3\alpha_6 - 2\alpha_3 - \alpha_0) - (2\alpha_8 - 3\alpha_9 + \alpha_{10}))E + ((4\alpha_3 + 3\alpha_1 - 7\alpha_4) + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10}))V + ((14\alpha_4 - \\
& 12\alpha_6 - 2\alpha_7) + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11}))H + ((8\alpha_0 + 4\alpha_7 - 12\alpha_9) + (8\alpha_{11} - 12\alpha_8 - 4)) \Big]
\end{aligned}$$

and

$$\begin{aligned}
\hat{F}_1 = & \left[((-12\alpha_6 + 8\alpha_0 + 4\alpha_3) + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8))A + (28\alpha_4 - 24\alpha_1 - 4\alpha_3) + (28\alpha_9 - 4\alpha_{11} - \right. \\
& 24\alpha_{10}))B + ((16\alpha_7 - 28\alpha_4 + 12\alpha_6) + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11}))C + ((24\alpha_1 - 8\alpha_0 - 16\alpha_7) + (24\alpha_8 - \\
& 16\alpha_9 - 8\alpha_{11}))D + ((12\alpha_6 - 8\alpha_0 - 4\alpha_3) + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8))E + ((28\alpha_4 - 24\alpha_1 - 4\alpha_3) + (28\alpha_9 - \\
& 4\alpha_{11} - 24\alpha_{10}))V + ((16\alpha_7 - 28\alpha_4 + 12\alpha_6) + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11}))H + ((24\alpha_1 - 8\alpha_0 - 16\alpha_7) + \\
& (24\alpha_8 - 16\alpha_9 - 8\alpha_{11}))G \Big].
\end{aligned}$$

Substituting equation (2.4) into (3.1) yields:

$$F = \Phi - \frac{\hat{F}_0}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3} \Phi - \frac{\hat{F}_1}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3} \hat{F}\Phi, \implies \Phi = [I - \hat{F}_2 - \hat{F}_3]^{-1} F, \quad (3.2)$$

where $\hat{F}_2 = \frac{\hat{F}_0}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3}$ and $\hat{F}_3 = \frac{\hat{F}_1}{2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3} \hat{F}$.

Finally, the exact solution φ is approximated by the exponential spline function

$\hat{S} = \hat{S}_j$, where

$$\begin{aligned}
\hat{S}_j(t) = & \left[\left(\frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)\hat{\varphi}_j - (2\alpha_8 - 3\alpha_9 + \alpha_{10})\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \right. \\
& \left. \left. - \frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)\hat{M}_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\lambda\tau(t-t_j)} \right. \\
& + \left(\frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\hat{\varphi}_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \\
& \left. + \frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)\hat{M}_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{2}}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\hat{\varphi}_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \\
& + \left. \frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)\hat{M}_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{4}} \\
& + \left(\frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\hat{\varphi}_j + (8\alpha_{11} - 12\alpha_8 + 4)\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \\
& \left. + \frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)\hat{M}_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{8}} \Big] + O(h^4). \tag{3.3}
\end{aligned}$$

Then

$$\begin{aligned}
\hat{e} = S_j(t) - \hat{S}_j(t) = & \left[\left(\frac{(3\alpha_6 - 2\alpha_3 - \alpha_0)(\varphi_j - \hat{\varphi}_j) - (2\alpha_8 - 3\alpha_9 + \alpha_{10})(\varphi_{j+1} - \hat{\varphi}_{j+1})}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \right. \\
& \left. - \frac{(12\alpha_6 - 8\alpha_0 - 4\alpha_3)\hat{M}_j + (12\alpha_9 - 8\alpha_{10} - 4\alpha_8)\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\lambda\tau(t-t_j)} \\
& + \left(\frac{(4\alpha_3 + 3\alpha_1 - 7\alpha_4)\hat{\varphi}_j + (4\alpha_{11} - 7\alpha_9 + 3\alpha_{10})\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \\
& \left. + \frac{(28\alpha_4 - 24\alpha_1 - 4\alpha_3)\hat{M}_j + (28\alpha_9 - 4\alpha_{11} - 24\alpha_{10})\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{2}} \\
& + \left(\frac{(14\alpha_4 - 12\alpha_6 - 2\alpha_7)\hat{\varphi}_j + (14\alpha_8 - 2\alpha_{10} - 12\alpha_{11})\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \\
& \left. + \frac{(16\alpha_7 - 28\alpha_4 + 12\alpha_6)\hat{M}_j + (16\alpha_{10} - 28\alpha_8 + 12\alpha_{11})\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{4}} \\
& + \left(\frac{(8\alpha_0 + 4\alpha_7 - 12\alpha_9)\hat{\varphi}_j + (8\alpha_{11} - 12\alpha_8 + 4)\hat{\varphi}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)} \right. \\
& \left. + \frac{(24\alpha_1 - 8\alpha_0 - 16\alpha_7)\hat{M}_j + (24\alpha_8 - 16\alpha_9 - 8\alpha_{11})\hat{M}_{j+1}}{(2\alpha_7 - 9\alpha_1 + 7\alpha_4 + 2\alpha_3)\lambda\tau} \right) e^{\frac{\lambda\tau(t-t_j)}{8}} \Big] + O(h^4),
\end{aligned}$$

Hence, the error estimate is given by

$$|\hat{e}| \leq \sigma_0 h^4, \tag{3.4}$$

where σ_0 is a constant.

4 Error estimation

Lemma 4.1 ([5]). Assume B is a square matrix with $\|B\| < 1$. Then, the matrix $(I - B)$ is invertible, and $\|(I - B)^{-1}\|_\infty \leq \frac{1}{1 - \|B\|_\infty}$.

Lemma 4.2. The matrix $[I - \hat{F}_2 - \hat{F}_3]$ in equation (3.2) is invertible if:

$$\|\psi\|_\infty (a_1 - a_0) \left(\sqrt{\frac{h}{\pi}} + \gamma_0 h^3 \right) < 1.$$

Assuming $\varphi \in C^4[I]$ and $\psi \in C^4[I \times I]$, we obtain:

$$\|\hat{e}\|_\infty \leq \sigma h^4,$$

where σ is a constant.

Proof. For $j = 0, 1, \dots, n$, we have:

$$\begin{aligned} \|A\|_\infty &= \|E\|_\infty \leq \|\psi\|_\infty(a_1 - a_0)h, \\ \|B\|_\infty &= \|V\|_\infty \leq \|\psi\|_\infty(a_1 - a_0)4e^{\lambda\tau h^{3/2}}, \\ \|C\|_\infty &= \|R\|_\infty \leq \|\psi\|_\infty(a_1 - a_0)|e^{\lambda\tau h^{5/2}} - 1|, \\ \|D\|_\infty &= \|G\|_\infty \leq \|\psi\|_\infty(a_1 - a_0)|1 - e^{\lambda\tau\sqrt{h}}|, \\ \|\hat{F}_2\|_\infty &\leq \|\psi\|_\infty(a_1 - a_0), \\ \|\hat{F}_3\|_\infty &\leq \|\psi\|_\infty(a_1 - a_0)e^{\lambda\tau h}, \\ \|M\|_\infty &\leq \|\psi\|_\infty(a_1 - a_0). \end{aligned}$$

By Lemma 1, equation (3.2) has a unique solution Φ . Equation (3.4) defines a unique solution \hat{S}_j , and we obtain:

$$\|\psi\|_\infty(a_1 - a_0) \left(\sqrt{\frac{h}{\pi}} + \gamma_0 h^3 \right) < 1,$$

and

$$\|e\|_\infty \leq \sigma h^4,$$

where γ_0 is a product of the matrices A^{-1} and B , $\sigma = \frac{\sigma_0}{\|\psi\|_\infty(a_1 - a_0) \left(\sqrt{\frac{h}{\pi}} + \gamma_0 h^3 \right)}$, and σ is a constant. □

5 Numerical results

In this section, three examples are presented to illustrate the accuracy of the method. The least-squares error (L.S.E.) is used:

$$\sum_{j=0}^M [\varphi(z_j) - \hat{\varphi}_N(z_j)]^2,$$

where M is a natural number. All computations are performed using Python with $N = 3$, $[a_0, a_1] = [0, 1]$, $\lambda = 10.9$, and $\tau = 0.060920$. State the purpose of Examples 5.1, 5.2, and 5.3: to test the Exponential Spline Function (ESF) method on the given problem (a specific integral equation defined by the preceding matrix equations and functions). Confirm that the numerical solution obtained by the ESF method will be compared against the exact solution.

Example 5.1 ([2]).

$$\begin{aligned} \varphi(z) &= f(z) + \lambda_1 \int_{a_0}^{a_1} \psi(z, t)\varphi(t) dt + \lambda_2 \int_{a_0}^z \psi(z, t)\varphi(t) dt, \\ \text{where } f(z) &= \frac{2}{3}z - \frac{1}{3}z^4, \lambda_{1,2} = 1, \psi(z, t) = zt, \text{ and } \varphi(z) = z. \end{aligned}$$

From the example consideration, the following constants and equations have been found:
 $z = [0, 0.33333333, 0.66666667, 1]$,
 $\alpha_0 = 1.1805813657639121, \alpha_1 = 1.318742107424998, \alpha_2 = 1.2137015634931758,$
 $\alpha_3 = 1.086545611451223, \alpha_4 = 1.28275550193187, \alpha_5 = 1.3557382862724932,$

$$\begin{aligned} \alpha_6 &= 1.1483649713505713, \alpha_7 = 1.3937723611889843, \alpha_8 = 1.117027716739953, \\ \alpha_9 &= 1.0568953196698114, \alpha_{10} = 1.028054142382497, \alpha_{11} = 1.2477509199652725, \\ \beta_0 &= 0.9203479259967465, \beta_1 = 0.9461674977540951, \beta_2 = 0.8708032942035129, \\ \beta_3 &= 0.9727114154537795, \beta_4 = 0.8470403048865129, \beta_5 = 0.8239257739125609, \\ \beta_6 &= 0.8952329338062456, \delta_0 = -0.0031204734921743693, \delta_1 = 0.0002075484925732063, \\ \delta_2 &= -0.21475300366307382, \gamma_0 = 0.2123534639495812, \gamma_1 = 30.916508342344425, \\ \gamma_2 &= 10.100432045686839. \end{aligned}$$

Equation (2.4) became:

$$(-0.0031204734921743693)M_{j-1} + (0.0002075484925732063)M_j + (-0.21475300366307382)M_{j+1} = (0.060920)[(0.2123534639495812)\varphi_{j-1} + (30.916508342344425)\varphi_j + (10.100432045686839)\varphi_{j+1}].$$

Then this system has been solved to obtain the M_j 's values:

$$\begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3.33500116e + 05 \\ -3.25937066e + 02 \\ 4.83909718e + 03 \\ 0 \end{bmatrix}.$$

Also

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -2376.36772965 \\ -2537.1627293 \\ 31.99263033 \\ 36.76941649 \end{bmatrix}, \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 17273.02091979 \\ 19133.92868832 \\ -231.86574973 \\ -277.30606547 \end{bmatrix},$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -35184.48865184 \\ -39700.44412914 \\ 471.59136581 \\ 575.38541876 \end{bmatrix}, \quad \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 20287.80799257 \\ 23104.00249541 \\ -271.04212668 \\ -333.82085563 \end{bmatrix}.$$

Then, all previous values have been put in equation (2.1) to obtain:

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_4 \end{bmatrix} = \begin{bmatrix} -2376e^{0.664(z-0)} + 17273e^{0.332(z-0.33333)} - 35185e^{0.166(z-0.6667)} + 20288e^{0.0830(z-1)} \\ -2537.2e^{0.664(z-0)} + 19134e^{0.332(z-0.33333)} - 39700e^{0.166(z-0.6667)} + 23104e^{0.0830(z-1)} \\ 31.993e^{0.664(z-0)} - 231.8657e^{0.332(z-0.33333)} + 471.5913e^{0.166(z-0.6667)} - 271.0421e^{0.0830(z-1)} \\ 36.76941e^{0.664(z-0)} - 277.3060e^{0.332(z-0.33333)} + 575.3854e^{0.166(z-0.6667)} - 333.8209e^{0.0830(z-1)} \end{bmatrix}.$$

So, S_j values have been placed in equation (3.1) to obtain:

$$\begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.29904978412519906 \\ 0.7049880963155819 \\ 0.9245701119914043 \end{bmatrix}.$$

Table 5.1 compares the exact solution with the ESF approximation using only $N = 3$ grid points. The absolute errors 3.4×10^{-2} , 3.8×10^{-2} , and 7.5×10^{-2} are notably small, and the leastsquares error is merely 8×10^{-3} . Such low errors, obtained with such a coarse mesh, attest to the high precision and efficiency of the ESF scheme. When the number of grid points is increased to $N = 10$ (Figure 5.1), the ESF and exact curves become visually indistinguishable, confirming the method's rapid convergence and robustness even with minimal discretization.

| z | Exact solution | ESF method | Absolute error |
|-----------------------------|----------------|------------|----------------------|
| 0 | 0 | 0 | 0 |
| 0.33333333 | 0.33333333 | 0.29904978 | 3.4×10^{-2} |
| 0.66666667 | 0.66666667 | 0.70498810 | 3.8×10^{-2} |
| 1 | 1 | 0.92457011 | 7.5×10^{-2} |
| L.S.E. = 8×10^{-3} | | | |

Table 5.1: Comparison between exact solutions and ESF method with $N = 3$ for Example 5.1.

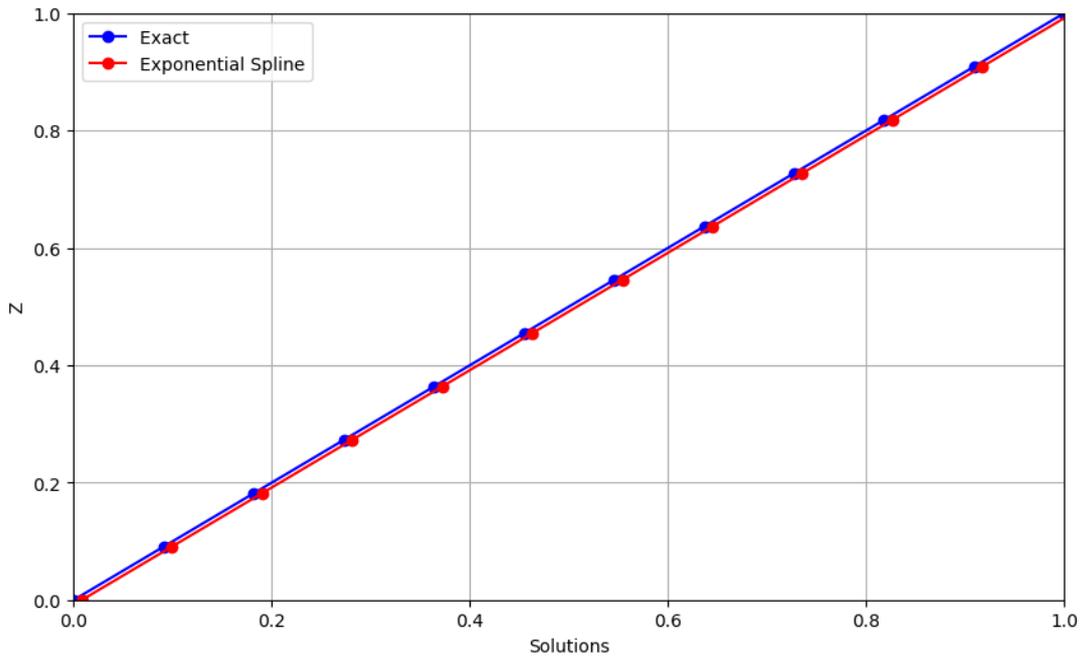


Figure 5.1: Comparison of the exact solution and exponential spline method results with $N = 10$ for Example 5.1.

Example 5.2 ([13]).

$$\varphi(z) = f(z) + \lambda_1 \int_{a_0}^{a_1} \psi(z,t)\varphi(t) dt + \lambda_2 \int_{a_0}^z \psi(z,t)\varphi(t) dt,$$

where $f(z) = z^2 - \frac{1}{12}z^4 - \frac{1}{3}z - \frac{1}{4}$, $\lambda_{1,2} = 1$,

$$\psi_V(z,t) = z - t, \psi_F(z,t) = z + t, \text{ and } \varphi(z) = z^2.$$

The comparison for $N = 3$ is shown in Table 5.2, and for $N = 10$ in Figure 5.2.

Example 5.3 ([3]).

$$\varphi(z) = f(z) + \lambda_1 \int_{a_0}^{a_1} \psi(z,t)\varphi(t) dt + \lambda_2 \int_{a_0}^z \psi(z,t)\varphi(t) dt,$$

where $f(z) = \cos(z) \left(\frac{1}{2}z - \frac{1}{4} \right) + \frac{1}{4} \cos(2 - z)$, $\lambda_{1,2} = 1$,

$$\psi_V(z,t) = \sin(z - t), \psi_F(z,t) = \cos(z - t), \text{ and } \varphi(z) = \sin(z).$$

The comparison for $N = 3$ is shown in Table 5.3, and for $N = 10$ in Figure 5.3.

| z | Exact solution | ESF method | Absolute error |
|-----------------------------|----------------|------------|----------------------|
| 0 | 0 | 0.25 | 2.5×10^{-1} |
| 0.11111111 | 0.01234568 | 0.11989119 | 1.1×10^{-1} |
| 0.44444444 | 0.19753086 | 0.29144929 | 9.4×10^{-2} |
| 1 | 1 | 1.09994527 | 9.9×10^{-2} |
| L.S.E. = 9×10^{-2} | | | |

Table 5.2: Comparison between exact solutions and ESF method with $N = 3$ for Example 5.2.

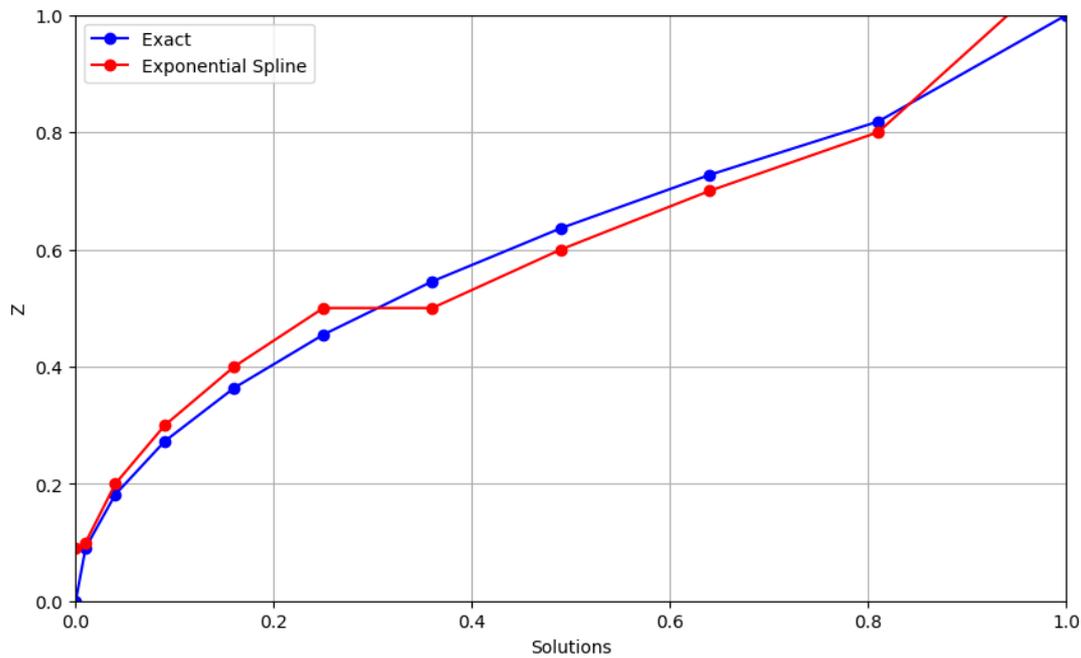


Figure 5.2: Comparison of the exact solution and exponential spline method results with $N = 10$ for Example 5.2.

| z | Exact solution | ESF method | Absolute error |
|-----------------------------|----------------|------------|----------------------|
| 0 | 0 | 0.35 | 3.5×10^{-1} |
| 0.5 | 0.47942554 | 0.49389696 | 1.4×10^{-2} |
| 0.75 | 0.68163876 | 0.73903126 | 5.7×10^{-2} |
| 1 | 0.84147098 | 0.89399123 | 5.2×10^{-2} |
| L.S.E. = 1×10^{-1} | | | |

Table 5.3: Comparison between exact solutions and ESF method with $N = 3$ for Example 5.3.

| L.S.E. | Example 5.1 | Example 5.2 | Example 5.3 |
|-------------|--------------------|--------------------|--------------------|
| ESF method | 8×10^{-3} | 9×10^{-2} | 1×10^{-1} |
| In [2,3,13] | 1×10^{-1} | – | 0.9 |

Table 5.4: Comparison between ESF method and previous methods for least squares error with $N = 3$.

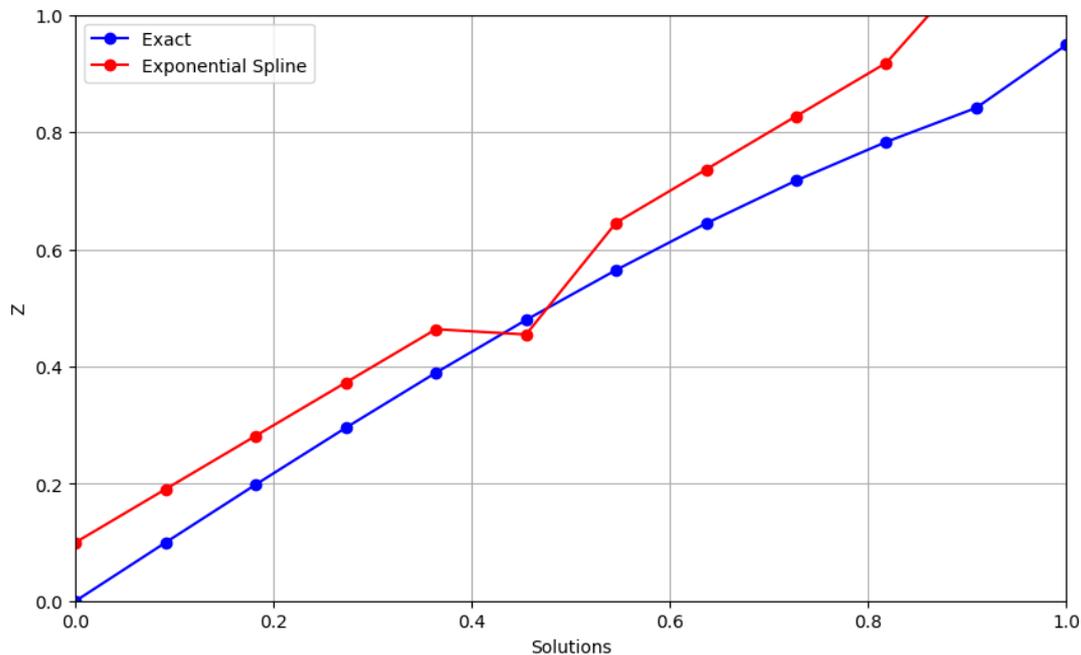


Figure 5.3: Comparison of the exact solution and exponential spline method results with $N = 10$ for Example 5.3.

6 Conclusion

This study introduces a novel numerical technique for solving linear Volterra-Fredholm integral equations of the second kind. The proposed method utilizes an innovative application of the exponential spline function (ESF). To establish the robustness and accuracy of this approach, several essential lemmas and theorems have been rigorously proven, supporting a comprehensive convergence analysis. Furthermore, a dedicated Python program was developed to generate and visualize both the exact and approximate solutions. These results, presented in detailed tables and figures, demonstrate that the proposed method offers superior performance compared to existing approaches in the field.

Declarations

Availability of data and materials

The data generated during this study are available from the corresponding author upon reasonable request.

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Not applicable.

Authors' contributions

K.D.: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing - original draft, Visualization.

M.M.: Methodology, Software, Validation, Writing- review & editing.

Conflict of interest

The authors have no conflicts of interest to declare.

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